THE MATHEMATICS OF PRODUCTION AND CONSUMPTION IN A STATIC ECONOMY

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PREFACE

Mathematical analysis as applied to economics is a relatively new science. As is the case in all new sciences, errors and misconceptions occur in the literature, controversies arise as to methods, and principles are slow to become established. Nowhere in the literature have I been able to find a complete and correct mathematical treatment of the general case of production and consumption.

Mathematics when correctly used provides the economist with a powerful and versatile tool, but the rules governing its use are rigid. If the economist is to support his argument with mathematical methods, he must adhere strictly to the mathematical rules.

It is not my purpose to condemn or to defend any economic theory, but rather to develop a sound basis for a mathematical approach to the simplest case; i.e., the static economy. In so doing, I have sacrificed some economic reality in order to obtain a solution and to retain mathematical rigor.

I have divided the development into three major parts; namely, the mathematical theory, certain applications to economics, and a mathematical model of a static economy. This approach has necessitated the inclusion of some

material already published. However, this material is scattered, generally incomplete, and fraught with errors. I have attempted to organize the essentials of the mathematics and the major portion of the applications to economics in the first two parts of the work. The nature of the material is such that it has been difficult to assign credit, although I have done so wherever possible.

I have made several references to "Unpublished Notes on Econometrics," University of Florida. These notes are in the custody of Dr. C. G. Phipps of the Mathematics Department of the University of Florida. His lectures and papers on the subject comprise the major portion of these notes, with contributions being made by Dr. G. B. Lang, Dr. M. D. Anderson, Mr. Ernest Lytle, Mr. R. N. Conkling, Mr. H. E. Whitsett and the author.

In the first two parts of this work I have drawn heavily upon the unpublished notes mentioned above. This was necessary in order to provide a correct background for the mathematical model of the last part.

I wish to express my appreciation to the original members of the committee supervising my graduate work, Dr. M. D. Anderson, Dr. O. E. Heskin, Dr. F. W. Kokomoor,

Dr. G. B. Lang; and to Dr. R. H. Blodgett who later substituted for Dr. Heskin. I am especially indebted to Dr. C. G. Phipps, the chairman of my committee. His constructive guidance and remarkable insight into the problem were of incalculable assistance in the preparation of this dissertation.

W. G. M.

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PART I

MATHEMATICAL THEORY

CHAPTER 1

MAXIMA AND MINIMA

Many years ago Alfred Marshall wrote, "... Experience in handling physical problems by mathematical methods gives a grasp, that cannot be obtained equally well in any other way, of the mutual interaction of economic changes ..." Among the methods to which he had reference are those relating to maximum and minimum values. Most of the problems which will be encountered later will in some way involve the maximum or minimum of some function; hence the first chapter in the mathematical theory will be devoted to a study of this topic.

The general approach will be a consideration of the necessary conditions followed by an investigation of the sufficient conditions. The developments will begin with functions of one independent variable and will be extended to functions of n independent variables. Special cases of functions of two and three independent variables will be included in order to clarify the arguments by means of geometric interpretations. In the latter portion of the chapter certain restraining conditions will be considered.

A. Necessary Conditions for Maxima and Minima.

Consider a function of one independent variable, $y = f(x)$, where $f(x)$, $f'(x)$ and $f''(x)$ are continuous.

We establish the following definition:

Definition 1:1:1. Maxima and Minima.

> The function $f(x)$ has a maximum (minimum) value at a point on $y = f(x)$ where the value of $f(x)$ is greater (less) than all other values in the immediate neighborhood of that point.*

Thus,

for a max. y_o, $f(x)-f(x_o) < 0$ in (x_o+e, x_o-e), $e \neq 0$;

for a min. y_o, $f(x)-f(x_o) > 0$ in (x_o+e, x_o-e), $e \neq 0$.

With continuity assumed through the second derivative, we may express $f(x)$ in a Taylor's series, with remainder, about x_o; i.e.,

$$(1:1:1) \quad y = f(x_o) + (x-x_o)f'(x_o) + \frac{(x-x_o)^2}{2!}f''\left[x_o+\theta(x-x_o)\right], \quad 0 < \theta < 1.$$

Since $f''(x)$ is continuous, it must have a

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* It should be noted that a maximum value of a function is not necessarily the "greatest" value of the function. A maximum value is greater than all neighboring values but there may be other greater values at points outside the neighborhood. The same observation holds for a minimum.

greatest absolute value in the closed interval (x_0+e, x_0-e). Let this greatest value be designated by M, so that $M = \max_{\circ}|f''(x)|$. We form the inequality

$$(1:1:2) \qquad |x - x_0| < \frac{2\, f'(x_0)}{M},$$

and then if $f'(x_0) \neq 0$, we have

$$(1:1:3) \qquad \left|(x - x_0)f'(x_0)\right| > \left|\frac{(x-x_0)^2}{2!}\, f''\left[x_0+\theta(x-x_0)\right]\right| .$$

The difference $y - f(x_0)$ must be negative for all values of x in the immediate neighborhood of x_0 for a maximum $f(x)$, and positive for a minimum $f(x)$. From (1:1:1) we have

$$(1:1:4) \qquad y-f(x_0) = (x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}\, f''\left[x_0+\theta(x-x_0)\right] .$$

Examining the right hand side of (1:1:4) in the light of the inequality (1:1:3), we observe that the difference $y-f(x_0)$ will have the sign of $(x-x_0)$ if $f'(x_0)$ is positive, and the sign opposite that of $(x-x_0)$ if $f'(x_0)$ is negative. In other words, the difference will be positive on one side of x_0 and negative on the other. Hence $y-f(x_0)$ can be made either positive or negative.

The only possible condition under which this situation can be prevented is if and only if $f'(x_0) = 0$.

We may, then, state

Theorem 1:1:1. For $f(x)$ to attain a maximum or minimum value at $x = x_0$ it is necessary that $f'(x_0) = 0$.*

We now consider the case of a function of two independent variables, $z = f(x,y)$, where the function and all of its derivatives through the second are continuous.

The function will attain a maximum value at $x = x_0$, $y = y_0$ when $f(x,y) - f(x_0,y_0)$ is negative for all values of x and y in the immediate neighborhood of (x_0,y_0), and will attain a minimum value when the difference is positive in that neighborhood.

We obtain a Taylor's expansion for $f(x,y)$ by defining a function $\phi(t)$ as

$$\phi(t) = f(x_0+at, y_0+bt). \tag{1:1:5}$$

Since continuity has been assumed through the second derivatives we may expand $\phi(t)$ in a Maclaurin's series as

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2!}\phi''(\theta t), \quad 0 < \theta < 1. \tag{1:1:6}$$

Differentiating, we have

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*If $y = x^3$, $f'(0) = 0$, but neither a maximum nor a minimum exists there. If $y = x^4$, $f'(0) = 0$, and a minimum does exist there. This example demonstrates that the condition of the theorem is not sufficient.

$$\phi'(t) = af_x(x_o+at,y_o+bt) + bf_y(x_o+at,y_o+bt), \text{ and}$$
$$\phi''(t) = a^2f_{xx}(x_o+at,y_o+bt) + 2abf_{xy}(x_o+at,y_o+bt) + b^2f_{yy}(x_o+at,y_o+bt).$$

Thus

$$\phi(0) = f(x_o,y_o),$$
$$\phi'(0) = af_x(x_o,y_o) + bf_y(x_o,y_o), \text{ and}$$
$$\phi''(0) = a^2f_{xx}(x_o+\theta at,y_o+\theta bt) + 2abf_{xy}(x_o+\theta at,y_o+\theta bt) + b^2f_{yy}(x_o+\theta at,y_o+\theta bt).$$

Setting $t = 1$, we have

$$f(x_o+a,y_o+b) = f(x_o,y_o) + \left[af_x(x_o,y_o) + bf_y(x_o,y_o)\right] + \frac{1}{2!}\left[a^2f_{xx}(x_o+\theta a,y_o+\theta b) + 2abf_{xy}(x_o+\theta a, y_o+\theta b) + b^2f_{yy}(x_o+\theta a,y_o+\theta b)\right].$$

Setting $x_o+a = x$ (or $a = x-x_o$) and $y_o+b = y$ (or $b = y-y_o$), we have

$$(1.1.7)\quad f(x,y) = f(x_o,y_o) + \left[(x-x_o)f_x(x_o,y_o)+(y-y_o)f_y(x_o,y_o)\right] + \frac{1}{2!}\Big\{(x-x_o)^2f_{xx}\left[x_o+\theta(x-x_o),y_o+\theta(y-y_o)\right] + 2(x-x_o)(y-y_o)f_{xy}\left[x_o+\theta(x-x_o),y_o+\theta(y-y_o)\right] + (y-y_o)^2f_{yy}\left[x_o+\theta(x-x_o),y_o+\theta(y-y_o)\right]\Big\}.$$

In order for a maximum value of $f(x,y)$ to exist it is necessary that $f(x,y) - f(x_o,y_o)$ be < 0 in the

immediate neighborhood where not both $x = x_o$ and $y = y_o$, that is, where $(x-x_o)^2 + (y-y_o)^2 \neq 0$.

If $x = x_o$ the argument of Theorem 1:1:1 will apply and $f_y(x_o,y_o)$ must equal zero.

An identical argument will hold for the case where $y = y_o$, and $f_x(x_o,y_o)$ must equal zero.

The necessary conditions for minimum are obtained by reversing the inequality signs.

We may, then, state

Theorem 1:1:2. For $z = f(x,y)$ to attain a maximum or a minimum value at (x_o,y_o) it is necessary that $f_x(x_o,y_o) = f_y(x_o,y_o) = 0$.

As in Theorem 1:1:1, the condition is not sufficient.

We now consider a function of n independent variables, $z = f(x_1,\ldots,x_n)$, and will develop the conditions necessary for z to attain a maximum or a minimum value at a point $(a_1,\ldots,a_n)$.*

We assume that the n independent arguments can be made differentiable functions of a single parameter, t, such that for $t = 0$, $x_k = a_k$, $k = 1,\ldots,n$, where $x_1 = x_1(t)$, ..., $x_n = x_n(t)$.

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*Based on a suggestion from Theodore Chaundy: "The Differential Calculus," pages 234-235.

For values of t such that $0 \leq |t| \leq e$ (where e is finite) the values of x_k will lie in the neighborhood of $(a_1,\ldots,a_n)$.

We now have the function

(1:1:8) $$z = f\left[x_1(t), \ldots, x_n(t)\right] = F(t).$$

The above equation represents a curve in a space of $n + 1$ dimensions, but since it contains only two variables we may examine it in a space of two dimensions.

For a particular choice of the functions, $x_1 = x_1(t)$, $\ldots$, $x_n = x_n(t)$, the values which z takes along the curve will form a sub-set of the values which z takes in the neighborhood of $(a_1,\ldots,a_n)$, and for a sufficiently small value of e, the value for $t = 0$ will be the minimum of this sub-set. In other words, z will be among the values which the function $f(x_1,\ldots,x_n)$ may take on in the neighborhood of $(a_1,\ldots,a_n)$. For a sufficiently small value of e, the value for $t = 0$ will be the minimum of this set of values.

Geometrically, $z = F(t)$ represents a curve in a space of $n + 1$ dimensions lying on the surface $z = f(x_1,\ldots,x_n)$ in that space. However, the curve has a minimum at $t = 0$, and since its equation contains only two variables, we may apply Theorem 1:1:1.

Thus, $\frac{dz}{dt}$ must equal zero for all choices of the

functions defining the x_k. Denoting the partials with respect to x_1 by f_1, etc., we have

(1:1:9) $\frac{dz}{dt} = f_1 x_1'(t) + \ldots + f_n x_n'(t) = 0$ for $t = 0$.

The above expression must be equal to zero for every choice of the $x_k'(t)$, hence the coefficients of the quantities must be zero at the minimum point. If $(a_1,\ldots,a_n)$ is a minimum point of the function it must be a minimum point for all possible curves through this minimum point. Similar argument holds for a maximum.

We may, then, state

Theorem 1:1:3. For $z = f(x_1,\ldots,x_n)$ to attain a maximum or a minimum value at $(a_1,\ldots,a_n)$ it is necessary that $f_k(a_1,\ldots,a_n) = 0$ for $k = 1,\ldots,n$.

As in the two preceding theorems, the condition is not sufficient.

B. Sufficient Conditions for Maxima and Minima.

For a function of one independent variable we are able to obtain the sufficient conditions for maxima and minima from (1:1:4). Making the assumption that $f'(x_o) = 0$, we have

(1:1:10) $y - f(x_o) = \frac{(x-x_o)^2}{2!} f''\left[x_o + \theta(x-x_o)\right]$.

Since $y-f(x_0)$ must be negative for $f(x)$ to attain a maximum value at $x = x_0$, and since the quantity $\frac{(x-x_0)^2}{2!}$ is always positive (except at $x = x_0$, where it can at most be zero), we may state

Theorem 1:1:4. If there exists a deleted neighborhood of x_0 such that $f''(x)$ is less than zero for every point of that neighborhood, then if $f'(x_0) = 0$, $f(x)$ will attain a maximum value at $x = x_0$.

Also, since $y-f(x_0)$ must be positive for $f(x)$ to attain a minimum value at $x = x_0$, we may conclude

Theorem 1:1:5. If there exists a deleted neighborhood of x_0 such that $f''(x)$ is greater than zero for every point of that neighborhood, then if $f'(x_0) = 0$, $f(x)$ will attain a minimum value at $x = x_0$.

The conditions of the two preceding theorems are not necessary. It is possible to construct a function having a maximum or a minimum at $x = x_0$ without satisfying the hypotheses of the theorems.

Determining the sufficient conditions for functions of more than one independent variable is considerably more difficult. In order to investigate these conditions we must make use of certain properties of positive

definite and negative definite quadratic forms. We will establish these properties as lemmas.

The general quadratic form is

$$(1:1:11)\qquad a_{11}x_1^2 + a_{22}x_2^2 + \ldots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \ldots + 2a_{23}x_2x_3 + \ldots ,$$

where $x_1,\ldots,x_n$ are the variables, and where the a's denote given coefficients such that the order of the subscripts is immaterial ($a_{12} = a_{21}$, etc.)

Definition 1:1:2. Definite Quadratic Forms.

> The quadratic form (1:1:11) is positive definite if it takes only positive values for all permissible values of the variables which are not all zero; and similarly, the quadratic form is negative definite if it takes only negative values.

We first consider the quadratic form in three variables,

$$(1:1:12)\qquad a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 .$$

Completing the square twice in succession, we have

$$a_{11}\left[x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right]^2 + \frac{a_{11}a_{22} - a_{12}^2}{a_{11}}x_2^2 + \frac{a_{11}a_{33} - a_{13}^2}{a_{11}}x_3^2 + 2\frac{a_{11}a_{23} - a_{13}a_{12}}{a_{11}}x_2x_3, \quad \text{or}$$

$$(1:1:13)\qquad a_{11}\left[x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right]^2 + \frac{a_{11}a_{22} - a_{12}^2}{a_{11}}\left[x_2 + \frac{a_{11}a_{23} - a_{13}a_{12}}{a_{11}a_{22} - a_{12}^2}x_3\right]^2 + \frac{a_{11}a_{22}a_{33}+2a_{12}a_{13}a_{23}-a_{11}a_{23}^2-a_{22}a_{13}^2-a_{33}a_{12}^2}{a_{11}a_{22}-a_{12}^2}\left[x_3\right]^2$$

Since the variables appear only within the brackets, and since each bracket is squared (and will therefore be positive for all real values of the variables), the original quadratic form (1:1:12) will be positive definite if, and only if, the coefficients of the brackets are positive, and it will be negative definite if, and only if, these coefficients are negative.

We may write the numerators of the coefficients of (1:1:13) in determinant form, as

$$a_{11} = a_{11}$$

$$(1:1:14)\qquad a_{11}a_{22} - a_{12}^2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}$$

$$a_{11}a_{22}a_{33}+2a_{12}a_{13}a_{23}-a_{11}a_{23}^2-a_{22}a_{13}^2-a_{33}a_{12}^2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

If the quadratic form (1:1:12) is to be positive definite, then all of the determinants of (1:1:14) must be positive. Similarly, if the quadratic form is to be

negative definite the determinants must be negative, positive, and negative respectively (since the denominators of the coefficients of the form will alternate in sign).

We may, then, state

Lemma 1:1:1. The quadratic form (1:1:12) is positive definite if, and only if, the determinants of (1:1:14) are all positive; and is negative definite if, and only if, these determinants are negative, positive, and negative respectively.

It should be noted that the above conditions are symmetrical and could have been written in two equivalent ways by electing to complete the square first on x_{22} or on x_{33}.

The method of completing the successive squares is perfectly general and we may extend the argument of Lemma 1:1:1 to the general case of the quadratic form (1:1:11). We have

Lemma 1:1:2. The quadratic form (1:1:11) is positive definite if, and only if,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0, \ldots, \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{vmatrix} > 0,$$

and is negative definite if, and only if, these expressions are alternately negative and positive.

We could have begun the process of completing squares with $a_{11}, a_{22}, \ldots, a_{nn}$. Thus all that we need is for one sequence of determinants to satisfy the conditions of the lemma.

We now consider the conditions under which a quadratic form is positive or negative definite, not for all values of the variables, but for values satisfying one or more linear relations in the variables. We shall not make use of these restricted definite forms until the next chapter is reached but we find it convenient to develop the conditions at this point.

Let the variables of the quadratic form (1:1:12) be related by the equation

(1:1:15) $$b_1x_1 + b_2x_2 + b_3x_3 = 0.$$

In order for (1:1:15) to constitute any restriction at all, at least one of the coefficients must be different from zero. Without loss of generality, we may take b_1 as $\neq 0$.

We have from (1:1:15)

$$x_1 = -\frac{1}{b_1}\left[b_2x_2 + b_3x_3\right],$$

and substituting in (1:1:12), we obtain

$$\frac{a_{11}}{b_1^2}(b_2x_2+b_3x_3)^2 + a_{22}x_2^2 + a_{33}x_3^2 - \frac{2a_{12}}{b_1}(b_2x_2+b_3x_3)x_2$$

$$- \frac{2a_{13}}{b_1}(b_2x_2+b_3x_3)x_3 + 2a_{23}x_2x_3,$$

which we may write as

(1:1:16) $\alpha x_2^2 + \beta x_3^2 + 2\gamma x_2 x_3$, where

$$\alpha = \frac{1}{b_1^2}(a_{22}b_1^2+a_{11}b_2^2-2a_{12}b_1b_2),$$

$$\beta = \frac{1}{b_1^2}(a_{33}b_1^2+a_{11}b_3^2-2a_{13}b_1b_3), \text{ and}$$

$$\gamma = \frac{1}{b_1^2}(a_{11}b_2b_3-a_{13}b_1b_2-a_{12}b_1b_3+a_{23}b_1^2).$$

By Lemma 1:1:2, the quadratic form (1:1:16) is positive definite if, and only if, $\alpha > 0$ and $\alpha\beta - \gamma^2 > 0$, or if

(1:1:17) $a_{22}b_1^2+a_{11}b_2^2-2a_{12}b_1b_2 > 0$, and

$$(a_{22}a_{33}-a_{23}^2)b_1^2+(a_{11}a_{33}-a_{13}^2)b_2^2+(a_{11}a_{22}-a_{12}^2)b_3^2$$
$$-2(a_{11}a_{23}-a_{13}a_{12})b_2b_3-2(a_{22}a_{13}-a_{23}a_{12})b_1b_3$$
$$-2(a_{33}a_{12}-a_{23}a_{13})b_1b_2 > 0.$$

The expressions of (1:1:17) can be written in determinant form having opposite signs from the expressions. Also, the quadratic form (1:1:16) will be negative definite if, and only if, $\alpha < 0$ and $\alpha\beta - \gamma^2 > 0$.

We may then, state

Lemma 1:1:3. The quadratic form (1:1:12) is positive definite subject to the linear constraint (1:1:15) if, and only if

$$\begin{vmatrix} 0 & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22} \end{vmatrix} < 0, \quad \begin{vmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & a_{11} & a_{12} & a_{13} \\ b_2 & a_{12} & a_{22} & a_{23} \\ b_3 & a_{13} & a_{23} & a_{33} \end{vmatrix} < 0,$$

and is negative definite subject to the constraint if, and only if, the two determinants are positive and negative respectively.

In comparing the conditions of Lemma 1:1:3 with those of Lemma 1:1:1 we note that the determinants of the former may be obtained by "bordering" those of the latter with the coefficients from the restricting relationship, and that the inequality relationships are reversed.

We may extend the proof to the general case by reducing the quadratic form from one in n variables to one in $n - 1$ variables by means of the constraining relationship. In this case we have

Lemma 1:1:4. The quadratic form (1:1:11) is positive definite subject to the relation

$$b_1x_1 + b_2x_2 + \ldots + b_nx_n = 0$$

if, and only if

$$\begin{vmatrix} 0 & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22} \end{vmatrix} < 0, \ldots, \begin{vmatrix} 0 & b_1 & b_2 & \cdots & b_n \\ b_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} < 0,$$

and is negative definite subject to the constraint if, and only if, the determinants are alternately positive and negative.

We note that the restricting relationship is not perfectly general since the expression equals zero instead of a constant. For systematic results we have applied the restriction in this manner. However, if the restraining equation were not equal to zero, then one more variable could be introduced into the whole problem and the development followed as before.

Returning to the sufficient conditions for a maximum or minimum value of a function of more than one independent variable we need only to consider the general case where $z = f(x_1,\ldots,x_n)$. Differentiating (1:1:9) with respect to t, we have

$$(1:1:18)\quad \frac{d^2z}{dt^2} = \left[f_1x_1'(t) + \ldots + f_nx_n'(t)\right] + \left\{f_{11}\left[x_1'(t)\right]^2 + \ldots + f_{nn}\left[x_n'(t)\right]^2 + 2f_{12}\left[x_1'(t)\cdot x_2'(t)\right] + 2f_{13}\left[x_1'(t)\cdot x_3'(t)\right] + \ldots\right\}.$$

From Theorem 1:1:3, the quantity $\left[f_1x_1'(t) + \ldots + f_nx_n'(t)\right]$ must vanish in order for z to attain a maximum or minimum value. What is left in (1:1:18), that is, the quantity in the braces, is a quadratic form in the $x_k'(t)$. If the quadratic form is negative definite, then the

function will attain a maximum value for every choice of the functions $x_k(t)$ and for $t \neq 0$. In order for the quadratic form to be negative definite we have, applying Lemma 1:1:2, that the determinants

(1:1:19)
$$f_{11}, \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix}, \ldots, \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{12} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{1n} & f_{2n} & & f_{nn} \end{vmatrix}$$

must be alternately negative and positive. We may, then, state

Theorem 1:1:6. If there exists a deleted neighborhood of $(a_1,\ldots,a_n)$ such that the determinants (1:1:19) are alternately negative and positive for every point in that neighborhood, then if $f_1(a_1,\ldots,a_n) = f_2(a_1,\ldots,a_n) = \ldots = f_n(a_1,\ldots,a_n) = 0$, the function $z = f(x_1,\ldots,x_n)$ will attain a maximum value at $(a_1,\ldots,a_n)$.

Also, from Lemma 1:1:2, we have

Theorem 1:1:7. If there exists a deleted neighborhood of $(a_1,\ldots,a_n)$ such that the determinants (1:1:19) are all positive for every point in that neighborhood, then if $f_1(a_1,\ldots,a_n) = f_2(a_1,\ldots,a_n) = \ldots = f_n(a_1,\ldots,a_n) = 0$, the function $z = f(x_1,\ldots,x_n)$ will attain a minimum value at $(a_1,\ldots,a_n)$.

CHAPTER 2

RESTRICTED MAXIMA AND MINIMA

In economic problems restrictive conditions often occur; hence in this chapter we will consider the conditions pertaining to restricted maxima and minima. This means that the variables are no longer independent, but are related by subsidiary equations. We will first examine the simple case of a function of two variables subject to one constraint. The problem can be solved directly by eliminating the dependent variable or by eliminating the differential of the dependent variable. However, as more variables and more constraints are introduced these methods rapidly become unwieldy. In developing a method for the general case we will introduce the "Lagrange multipliers." This method is of tremendous value in the analysis of several major economic phenomena, and the chief portion of this chapter will be devoted to a detailed investigation of the use and significance of these multipliers.

Let $z = f(x,y)$ be a function whose maximum and minimum values are to be found with the restriction that the variables satisfy the subsidiary relation $\phi(x,y) = 0$.

In the simplest case the subsidiary relation will determine one of the variables as a single-valued

function of the other, say $y = g(x)$, and upon substituting we have $z = f(x,y) = F(x)$. The function $F(x)$ will appear in single-valued form and the problem is easily solved.

A more subtle, and more advantageous, approach is attained by taking x as the dependent variable and then eliminating dx.

Making the necessary assumptions regarding the derivatives, we have

(1:2:1) $$dz = f_x dx + f_y dy = 0,$$

since dz must be zero in order for z to attain a maximum or minimum value.

From the subsidiary relation we have

(1:2:2) $$\phi_x dx + \phi_y dy = 0.$$

Taking x as the dependent variable given by the subsidiary relation, and eliminating dx, we have

(1:2:3) $$dz = \left[f_y - f_x \frac{\phi_y}{\phi_x} \right] dy = 0$$

for all choices of dy. Therefore, the coefficient of dy is zero, and we have

(1:2:4) $$\frac{f_x}{\phi_x} = \frac{f_y}{\phi_y} .$$

The necessary condition of (1:2:4) together with the equation $\phi(x,y) = 0$ enables us to solve for x and y.

To obtain the sufficient conditions we have from (1:2:1)

(1:2:5) $d^2z = f_x d^2x + f_{xx}dx^2 + 2f_{xy}dx\,dy + f_{yy}dy^2,$

and from (1:2:2) we have

(1:2:6) $\phi_x d^2x + \phi_{xx}dx^2 + 2\phi_{xy}dx\,dy + \phi_{yy}dy^2 = 0.$

Eliminating d^2x, we have

(1:2:7) $$d^2z = \left[f_{xx} - \frac{f_x}{\phi_x}\phi_{xx}\right]dx^2 + 2\left[f_{xy} - \frac{f_x}{\phi_x}\phi_{xy}\right]dx\,dy + \left[f_{yy} - \frac{f_x}{\phi_x}\phi_{yy}\right]dy^2.$$

If d^2z is negative when $dz = 0$, then z will attain a maximum value, and if d^2z is positive when $dz = 0$, then z will attain a minimum value.

In order for (1:2:7) to be a negative definite quadratic form subject to the relation $\phi_x dx + \phi_y dy = 0$, we have from Lemma 1:1:4 that

(1:2:8) $$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & \left[f_{xx} - \frac{f_x}{\phi_x}\phi_{xx}\right] & \left[f_{xy} - \frac{f_x}{\phi_x}\phi_{xy}\right] \\ \phi_y & \left[f_{xy} - \frac{f_x}{\phi_x}\phi_{xy}\right] & \left[f_{yy} - \frac{f_x}{\phi_x}\phi_{yy}\right] \end{vmatrix} > 0\ ;$$

and in order for (1:2:7) to be a positive definite quadratic form the determinant (1:2:8) must be < 0.

At this point we find it advantageous to investigate the nature and possibilities of the Lagrange multipliers. Their use was first explained by J. L. Lagrange in his

"Théorie des Fonctions Analytiques." The principles which we shall outline here are completely general and may be extended to functions of any number of variables. With slight modifications, the principles can be applied where several subsidiary relations are imposed, provided the number of subsidiary relations is less than the number of variables involved. These extensions are dealt with in several texts on the calculus.*

In this discussion we will justify the usual manipulation of the Lagrange multipliers, since apparently there has been established no proof of the possibility of such manipulation. Also, we will give a geometric interpretation in addition to the analytic interpretation, and we will analyze the results of imposing more than one constraint.

Let $z = f(x,y)$ be a function whose maxima and minima are to be found with the restriction that the variables satisfy the subsidiary relation $\phi(x,y,a,c) = 0$. The quantities a and c are parameters whose significance will be discussed later. Here we take them as being fixed in value and for the present write the constraining equation as $\phi(x,y) = 0$.

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* See Theodore Chaundy, "The Differential Calculus," p. 256 ff.
R. Courant, "Differential and Integral Calculus," vol. ii, p. 188 ff.

We define the function u as

(1:2:9) $$u = f(x,y) - \lambda\phi(x,y).$$

The λ of the above equation is the Lagrange multiplier. It may be taken with either sign. We will define this quantity more precisely at a later point.

In practically every mathematics text λ is regarded as an unknown constant whose value could be found but does not need to be. In a few texts it is regarded as a function of x and y (Lagrange, in fact, so regarded λ). In texts on mathematical economics the multiplier is usually regarded as a function of the parameters a and c. Actually, however, it may be any one of the above.

We wish to examine the function u for maximum and minimum values. The equation $\phi(x,y) = 0$ may be regarded as defining a curve in the xy-plane. Taken along this curve, the variable u has a maximum or a minimum whenever z does, irrespective of the form or value of λ (as long as λ is defined and finite). This includes as a special case a constant value for λ.

Whenever u and z attain a maximum or minimum value at a common point we have

(1:2:10) $$du = (f_x - \lambda\phi_x)dx + (f_y - \lambda\phi_y)dy - \phi(\lambda_x dx + \lambda_y dy) = 0.$$

At the same point, because of the subsidiary

relation, dx and dy are connected by the equation

(1:2:11) $$d\phi = \phi_x dx + \phi_y dy = 0.$$

In (1:2:11) either dx or dy may be taken as dependent. Here we choose dy as being dependent.

We will now define λ more precisely. In (1:2:10) the last term contains the factor ϕ and therefore vanishes. Hence du can be made entirely independent of the dependent differential dy by the proper choice of λ. This choice defines λ as

(1:2:12) $$\lambda = \frac{f_y}{\phi_y}.$$

It should be noted that ϕ_y must not = 0.

Thus, λ may be taken as the numerical value assumed by the fraction of (1:2:12) at a maximum or minimum point if there is only one such point. If there is more than one maximum or minimum point, then λ must be taken as multi-valued, with a corresponding value at each maximum or minimum point. Consequently, it is more appropriate to regard λ as a function of x and y, since it will then automatically assume the correct value at each critical point.

However, with either definition, and the fact that du = 0 is a necessary condition for u to attain a maximum or minimum value, equation (1:2:10) reduces to

(1:2:13) $$du = (f_x - \lambda\phi_x)dx = 0,$$

and since this must hold for all values of the independent differential dx, the coefficient of dx must be zero.

We have, then,

Theorem 1:2:1. A necessary condition that the function $z = f(x,y)$ attain a maximum or minimum value subject to the subsidiary relation $\phi(x,y) = 0$ is that $\frac{f_x}{\phi_x} = \frac{f_y}{\phi_y} = \lambda$.

Several implications of the above theorem are worthy of note. First, if a derivative of ϕ is zero, then the corresponding derivative of f is also zero, but not conversely. Second, if both derivatives of ϕ are zero, then ϕ has a singular point which requires special investigation. Third, the subsidiary relation taken with the equation $\frac{f_x}{\phi_x} = \frac{f_y}{\phi_y}$ serves to determine the values of x and y at critical points without the necessity of evaluating λ.

A sufficient condition that u, and therefore z, attain a maximum value is that d^2u be less than zero when $du = 0$, and to attain a minimum value that d^2u be greater than zero when $du = 0$. For the second differential of u we have

$$(1:2:14)\quad \begin{aligned} d^2u = {} & (f_{xx}-\lambda\phi_{xx})dx^2+2(f_{xy}-\lambda\phi_{xy})dx\,dy \\ & +(f_{yy}-\lambda\phi_{yy})dy^2-2(\lambda_x dx+\lambda_y dy)(\phi_x dx+\phi_y dy) \\ & -\phi(\lambda_{xx}dx^2+2\lambda_{xy}dx\,dy+\lambda_{yy}dy^2) \\ & +(f_y-\lambda\phi_y-\phi\lambda_y)d^2y. \end{aligned}$$

In the above equation the coefficient of d^2y is zero because of the manner in which λ and ϕ have been defined. The next to last term is zero because it contains the factor ϕ. The third from last term is zero from (1:2:11). Hence (1:2:14) reduces to

(1:2:15) $$d^2u = (f_{xx} - \lambda\phi_{xx})dx^2 + 2(f_{xy} - \lambda\phi_{xy})dx\,dy + (f_{yy} - \lambda\phi_{yy})dy^2.$$

To establish the condition under which (1:2:15) will be negative definite or positive definite subject to the restriction $\phi_x dx + \phi_y dy = 0$, we have only to apply Lemma 1:1:4.

The preceding discussion also enables us to justify the usual methods of manipulating the Lagrange multiplier. We state these conclusions in the following lemma:

Lemma 1:2:1. With λ defined as a function of x and y by either of the equations $\lambda = \frac{f_x}{\phi_x}$ or $\lambda = \frac{f_y}{\phi_y}$, the forms of du and d^2u are the same (i) as if the multiplier, λ, were regarded as a constant, and (ii) as if both the variables, x and y, were regarded as independent.

The multiplier can be best interpreted by using the parameters of the restraining equation. For this purpose we now write ϕ as

(1:2:16) $$\phi(x,y,a) = c.$$

We modify the form of (1:2:9) to

(1:2:17) $u = f(x,y) - \lambda\left[\phi(x,y,a) - c\right]$.

In the above equation λ will have exactly the same properties as before. However, the values of x, y, and λ will differ for different values of c, since the equations $\lambda = \frac{f_x}{\phi_x} = \frac{f_y}{\phi_y}$ will be solved simultaneously with (1:2:16) instead of with $\phi(x,y) = 0$.

With the parameter a fixed, for each value of c there will be a point or set of points at which, or for the coordinates of which, $f(x,y)$ will attain a maximum or minimum value subject to the subsidiary relation $\phi(x,y,a) = c$. As c varies continuously, these points will, in general, trace a continuous path or paths in the xy-plane. Borrowing from the terminology of the economists, these paths may be identified as "expansion paths," for as c expands from zero to infinity, the critical points of $f(x,y)$ traverse these paths.

For each value of a, there will be a set of paths. Along any one path the parameter a is constant and the parameter c is variable.

In order to picture this in the xy-plane, we first construct in this plane the contour lines along which z is constant. We then draw a single curve of the family $\phi(x,y,a) = c$ by assigning particular values to a and c.

The slope of a contour line at a given point is

$\frac{dy}{dx} = -\frac{f_x}{f_y}$, where the partial derivatives are evaluated for the coordinates of the point. The slope at any point along the curve $\phi(x,y,a) = c$ is $\frac{dy}{dx} = -\frac{\phi_x}{\phi_y}$.

At a restricted maximum or minimum point of $f(x,y)$ these slopes are, by Theorem 1:2:1, equal. Therefore, at such a point, the curve $\phi(x,y,a) = c$ is tangent to a contour line of $f(x,y)$. This point of tangency is one point on the expansion path for the given value of the parameter a. Different values of c locate different points on the expansion curve.

If somewhere along the curve $\phi(x,y,a) = c$ the function $f(x,y)$ should attain an unrestricted maximum or minimum, then at that point both f_x and f_y are zero by Theorem 1:1:2, and, from Theorem 1:2:1, λ must also be zero. Hence, as a special case, we have the vanishing of λ as a necessary condition for $f(x,y)$ to attain an unrestricted maximum or minimum along the curve $\phi(x,y,a) = c$. In this case the points at which, or the contours along which, $f(x,y)$ has a critical value can be found by setting $f_x = f_y = 0$. If the critical value is attained at a point, and if this point is on the ϕ-curve, then it is the solution. If the critical value is attained along a contour line, the simultaneous solution of the equation of the contour line with the ϕ-curve will yield the solution.

We will now investigate the effects of changes in c and z along any one expansion path. The differential of c is $dc = \phi_x dx + \phi_y dy$, and at the same point along the same path the differential of z is $dz = f_x dx + f_y dy$. The ratio of dz to dc is

$$\frac{dz}{dc} = \frac{f_x dx + f_y dy}{\phi_x dx + \phi_y dy} . \tag{1:2:18}$$

From Theorem 1:2:1, $f_x = \lambda \phi_x$ and $f_y = \lambda \phi_y$, and (1:2:18) reduces to

$$\frac{dz}{dc} = \lambda . \tag{1:2:19}$$

Thus, if λ is positive, the critical values of z increase with c, and if λ is negative they decrease as c increases. The sign of λ also has an important bearing upon the interpretation of the results when the roles of f and ϕ are interchanged.

In the preceding discussion we have held c fixed and examined z for maximum and minimum values. We will now hold z fixed at z_o and examine c for maximum and minimum values.

We define a function v as

$$v = \phi(x,y) - \mu\left[f(x,y) - z_o\right] . \tag{1:2:20}$$

As before, we have that $\mu = \frac{\phi_x}{f_x} = \frac{\phi_y}{f_y}$, and comparing this with the previous results we obtain the relationship between the multipliers as $\mu = \frac{1}{\lambda}$.

Also, as before, we have

$$(1:2:21) \quad d^2v = (\phi_{xx} - \mu f_{xx})dx^2 + 2(\phi_{xy} - \mu f_{xy})dx\,dy + (\phi_{yy} - \mu f_{yy})dy^2.$$

A comparison of the above equation with the previous equation (1:2:14), together with the reciprocal relationship between the multipliers, enables us to reach the conclusion $d^2u = -\lambda d^2v$. From this conclusion we formulate

Lemma 1:2:2. (i) If λ is positive at a point where z attains a maximum value $z = z_o$ for a value of $c = c_o$, then at that point c_o is a minimum value for c, and conversely. (ii) If λ is negative, then c and z will attain maximum and minimum values together.

The geometric interpretation for the case of a function of three variables, $u = f(x,y,z)$, with one constraint, $\phi(x,y,z) = a$, can be developed with the aid of vector analysis. In this case the multiplier λ is the ratio of the gradient of $f(x,y,z)$ at a maximum or minimum point to the gradient of $\phi(x,y,z)$ at that point, or $\lambda = \frac{d}{da}$(maximum or minimum of f). The development will not be made here but can be found in "Unpublished Notes on Econometrics", Mathematics Department, University of Florida, 1950.

We will now consider the case of two subsidiary

relations.

Let $u = f(x,y,z)$ be a function whose maximum and minimum values are to be found with the restrictions that

(1:2:22) $$\phi(x,y,z) = a \ , \quad \text{and}$$

(1:2:23) $$\psi(x,y,z) = b \ ,$$

where a and b are parameters. We assume f, ϕ, and ψ to be continuous, together with their first partial derivatives, and that $\frac{\partial(\phi,\psi)}{\partial(y,z)} \neq 0$.

We define the function

(1:2:24) $$w = f(x,y,z) - \lambda\left[\phi(x,y,z)-a\right] - \mu\left[\psi(x,y,z)-b\right] \ ,$$

where λ and μ are quantities to be defined. Subject to the restrictions $\phi = a$ and $\psi = b$, u and w will attain maximum and minimum values together.

We have, from the necessary conditions for a maximum or a minimum, that

(1:2:25) $$\begin{aligned} dw = {} & (f_x - \lambda\phi_x - \mu\psi_x)dx + (f_y - \lambda\phi_y - \mu\psi_y)dy \\ & + (f_z - \lambda\phi_z - \mu\psi_z)dz \\ & - (\phi - a)d\lambda - (\psi - b)d\mu = 0. \end{aligned}$$

Noting that $(\phi - a)$ and $(\psi - b)$ are both zero, we write the above equation as

(1:2:26) $$\begin{aligned} dw = {} & (f_x - \lambda\phi_x - \mu\psi_x)dx + (f_y - \lambda\phi_y - \mu\psi_y)dy \\ & + (f_z - \lambda\phi_z - \mu\psi_z)dz = 0 \ . \end{aligned}$$

We may now define λ and μ by

(1:2:27) $f_y - \lambda\phi_y - \mu\psi_y = 0$, and

(1:2:28) $f_z - \lambda\phi_z - \mu\psi_z = 0$.

With the above definitions, (1:2:26) reduces to $dw = (f_x - \lambda\phi_x - \mu\psi_x)dx = 0$; and, since dx is arbitrary, we have that

(1:2:29) $f_x - \lambda\phi_x - \mu\psi_x = 0$.

The five equations (1:2:22), (1:2:23), (1:2:27), (1:2:28) and (1:2:29) can be solved for the values of x, y, z, λ and μ for which u attains a maximum or minimum value subject to the restrictions $\phi = a$ and $\psi = b$.

If we permit the parameters, a and b, to vary we will obtain different corresponding values for x, y, z, λ and μ. These last five variables will, at a maximum or minimum point, be functions of a and b. We denote these functions by $x = x(a,b)$, $y = y(a,b)$, $z = z(a,b)$, $\lambda = \lambda(a,b)$ and $\mu = \mu(a,b)$.

If M is a maximum or minimum value of u, then it is also a function of a and b. We denote this function by $M = f\left[x(a,b),\ y(a,b),\ z(a,b)\right]$.

Differentiating M partially with respect to a, we have

(1:2:30) $$\frac{\partial M}{\partial a} = f_x \frac{\partial x}{\partial a} + f_y \frac{\partial y}{\partial a} + f_z \frac{\partial z}{\partial a}.$$

From (1:2:27), (1:2:28) and (1:2:29), we have

$$f_x = \lambda\phi_x + \mu\psi_x$$
$$f_y = \lambda\phi_y + \mu\psi_y$$
$$f_z = \lambda\phi_z + \mu\psi_z \quad .$$

Substituting these values in (1:2:30), rearranging the terms and factoring, we obtain

(1:2:31) $$\frac{\partial M}{\partial a} = \lambda\left(\phi_x \frac{\partial x}{\partial a} + \phi_y \frac{\partial y}{\partial a} + \phi_z \frac{\partial z}{\partial a}\right) + \mu\left(\psi_x \frac{\partial x}{\partial a} + \psi_y \frac{\partial y}{\partial a} + \psi_z \frac{\partial z}{\partial a}\right).$$

Differentiating (1:2:22) and (1:2:23) partially with respect to a, we have

$$\phi_x \frac{\partial x}{\partial a} + \phi_y \frac{\partial y}{\partial a} + \phi_z \frac{\partial z}{\partial a} = 1, \quad \text{and}$$

$$\psi_x \frac{\partial x}{\partial a} + \psi_y \frac{\partial y}{\partial a} + \psi_z \frac{\partial z}{\partial a} = 0.$$

When the above values are substituted in (1:2:31) we obtain the relation $\frac{\partial M}{\partial a} = \lambda$.

In a similar manner we may establish that $\frac{\partial M}{\partial b} = \mu$.

Thus the multiplier λ measures the rate of change of the maximum or minimum values of u with respect to a change in the parameter a, and μ measures the rate of change of the maximum or minimum values of u with respect to a change in the parameter b.

CHAPTER 3

RELATED TOPICS

This chapter will deal with three topics related to the general problem; namely, Dependent and Independent Functions, Conditional and Identical Equations, and Homogeneous Equations. Many of the concepts of this chapter were used in the two previous chapters, but since these topics are of great importance throughout econometrics, and since many of the errors in the extant literature in that field can be traced to misuse or misunderstanding of them, they will be re-examined in more detail in this chapter. Even trained mathematicians are often careless in their writings on these topics and the confusion is compounded when those without mathematical training go on to commit errors which the trained mathematician would not make.

A. Dependent and Independent Functions.

The variables involved in the solution of any problem fall into three categories; namely, parameters, dependent variables and independent variables. The parameters are the literal quantities which must be given for a particular problem and which remain fixed throughout most of the solution. In certain types of problems it is desirable to investigate the effects of varying the parameters. The latter part of the last chapter exemplifies this type of

analysis. The dependent variables are those literal quantities whose values are determined by the conditions of the problem. In general, the number of dependent variables is equal to the number of independent equations. The independent variables are those literal quantities which remain after the parameters and dependent variables have been identified.

In order for a problem to have a solution, the number of independent variables cannot exceed the number of independent equations. Thus, it becomes important to be able to specify the dependency or independency of a set of functions.

Definition 1:3:1. Dependent Functions.

A set of functions is dependent if, when values are assigned to some of the functions of the set, the values of one or more of the other functions are determined.

Definition 1:3:2. Independent Functions.

A set of functions is independent if it is not a dependent set.

It should be noted that the two classifications above are mutually exclusive.

We now consider the set of n functions of n variables

$$
\begin{aligned}
&{}^{1}f(x_1, x_2, \ldots, x_n) \\
&{}^{2}f(x_1, x_2, \ldots, x_n) \\
&\quad\vdots \quad\vdots \quad\quad\vdots \\
&{}^{n}f(x_1, x_2, \ldots, x_n),
\end{aligned}
\tag{1:3:1}
$$

where the values assigned to the variables are from a certain region R. We have

Theorem 1:3:1. A necessary and sufficient condition for the set of functions of (1:3:1) to be dependent is that its Jacobian be identically zero.*

The Jacobian of the set is

$$(1:3:2) \qquad J = \begin{vmatrix} {}^1f_1 & {}^1f_2 & \cdots & {}^1f_n \\ {}^2f_1 & {}^2f_2 & \cdots & {}^2f_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ {}^nf_1 & {}^nf_2 & & {}^nf_n \end{vmatrix} ,$$

where ${}^if_j$ indicates the partial derivative of the i th function with respect to x_j.

In econometrics three situations arise in which Theorem 1:3:1 affords a means of identifying dependent sets of functions. We will state as corollaries to the above theorem the conditions of these situations.

If the elements of any row of (1:3:2) are zero, then one of the functions is a constant. Hence we have

Corollary 1:3:1-(i). If one or more of the functions of a set is constant, then the set of functions is a dependent set.

If all of the elements of a column of (1:3:2) can be made equal to zero by a linear combination of the

*See de la Vallée Poussin: "Cours d'Analyse Infinitésimale", p. 274.

elements of one or more of the other columns, then the function from that column is functionally dependent upon one or more of the other functions. Hence we have

Corollary 1:3:1-(ii). If at least one function of a set of functions is connected by a functional relationship to one or more of the other functions, then the set of functions is a dependent set.

If we introduce an additional variable, x_{n+1}, into the set of functions the Jacobian becomes

$$\begin{vmatrix} {}^1f_1 & {}^1f_2 & \cdots & {}^1f_n & {}^1f_{n+1} \\ {}^2f_1 & {}^2f_2 & \cdots & {}^2f_n & {}^2f_{n+1} \\ \cdot & \cdot & & \cdot & \cdot \\ {}^nf_1 & {}^nf_2 & \cdots & {}^nf_n & {}^nf_{n+1} \\ 0 & 0 & \cdots & 0 & 0 \end{vmatrix} ,$$

and is identically zero. Hence we have

Corollary 1:3:1-(iii). If the number of functions of a set exceeds the number of independent variables, then the set of functions is a dependent set.

We may assign to the set of functions of (1:3:1) values from the region S and obtain the equations

$$\begin{aligned} u_1 &= {}^1f(x_1, x_2, \ldots, x_n) \\ u_2 &= {}^2f(x_1, x_2, \ldots, x_n) \\ &\cdots \\ u_n &= {}^nf(x_1, x_2, \ldots, x_n) . \end{aligned} \qquad (1:3:3)$$

We may solve the above set of equations for $x_1, x_2, \ldots, x_n$ as

$$
\begin{aligned}
x_1 &= {}^1\phi(u_1, u_2, \ldots, u_n) \\
x_2 &= {}^2\phi(u_1, u_2, \ldots, u_n) \\
&\cdots\cdots\cdots\cdots \\
x_n &= {}^n\phi(u_1, u_2, \ldots, u_n)
\end{aligned}
\qquad (1:3:4)
$$

if, and only if, the Jacobian, $J\,\frac{(u_1, u_2, \ldots, u_n)}{(x_1, x_2, \ldots, x_n)}$, is different from zero.* The assumption following (1:2:23), that $\frac{\partial(\phi, \psi)}{\partial(x, y)} \neq 0$, is an example of the necessity of this condition.

If the set of functions of (1:3:1) is a dependent set, then the maximum number to which arbitrary values can be assigned in order to obtain the set of equations (1:3:3) is equal to the rank of the matrix. The particular functions to which values may be assigned must be those from which the matrix acquires its rank. If the remaining functions are assigned the values which their dependency requires, then the set of equations, taken as a whole, is consistent. If any of the remaining functions are assigned values which are different from those required by their dependency, then the set of equations, taken as a whole, is inconsistent.

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*See F. S. Woods: "Advanced Calculus," p. 101.

The above concepts of consistency and inconsistency of sets of equations are basic in the sense that they are described in terms of the sources of the equations. A grasp of these concepts is valuable in the formulation of economic problems as well as in their solutions.

If the set of functions of (1:3:1) is an independent set, and values are assigned to them as in (1:3:3), then the resulting set of equations is always an independent and consistent set of equations.

B. Conditional and Identical Equations.

Much of the mathematical, as well as the econometrical, literature is remiss in dealing with conditional and identical equations, chiefly through the misuse of the identity relationship. It is written many times when properly it should not be, and it is not written when greater clarity would be obtained by using it.

In this section we will first establish a set of precise definitions. The definitions themselves are well known and easily understood but the conclusions which are derived from them are not carefully observed in practice.

Definition 1:3:3. Arithmetical Equations.

> An arithmetical equation is a statement that two groups of arithmetical operations yield the same number.

One of the groups may be a single number, and in the trivial case both may be the same number.

In the field of analysis the above definition is not appropriate if a literal quantity appears among the indicated operations. In this event it is necessary to give two definitions to fit the two situations which may arise.

Definition 1:3:4. Conditional Equations.

> A conditional equation, or more simply an equation, is a statement that two groups of operations will yield the same number for certain values of the literal quantities involved.

Definition 1:3:5. Identical Equations.

> An identical equation, or more simply an identity, is a statement that two groups of operations will yield the same number for all permissible values of the literal quantities involved.

In an equation the restrictions in value upon some of the literal quantities, in order that the equation be true, form the conditions imposed by the equation. In an identity the only restrictions upon the values used are that the operations be defined for those values.

A single equation containing a single variable may

be satisfied by only one numerical quantity, a finite number of such quantities, or an infinite number of such quantities. An expression may have the appearance of an equation but unless it is satisfied by at least one value it is not properly an equation.

In conditional equations the number of literal quantities, or dependent variables, upon which restrictions are placed is, in general, exactly equal to the number of independent equations. The literal quantities in excess of this number become either independent variables or parameters.

In identical equations all of the literal quantities are independent variables (or parameters) so far as each identity is concerned. If one or more of these literal quantities are dependent variables when they appear in other equations of a set, the identities do not impose any additional conditions upon them. Thus an identity cannot be counted as one of a set of simultaneous equations to be solved for a set of dependent variables. The only use for an identity under such circumstances is to effect a simplification or transformation of some kind upon other equations.

Identical equations may originate in many ways, the simplest arising from the fundamental operations of algebra.

A second source lies in the relationships which

grow out of correct observations. For example, we observe that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, for all positive integral values of n. In fact, the left member of the expression is defined only for these special values. The correctness of the observation is demonstrated by mathematical induction. Consequently, this relationship is an identity for all values of n for which the operations are defined.

We note the contrast in the above type of restriction with that of Definition 1:3:4. In the case of an identical equation the restriction is that the operations be defined, and if they are defined, then the equation is true. In the case of a conditional equation the operations may be defined for infinitely many values of the variables involved, and the restriction imposed is that the values be limited to those for which the equation is found to be true.

A third source of identities lies in the elimination of the dependent variable from two or more equations, only one of which is independent. For example, the equation $x^2 + 2bx + b^2 = c^2$ has for one of its solutions $x = b + c$. If this value for x is substituted in the original equation we obtain $(b + c)^2 - 2b(b + c) + b^2 \equiv c^2$ as an identity in b and c. This is one method by which trigonometric identities are derived.

We will now consider the way in which identities and equations can be combined. The corresponding members of two conditional equations cannot legitimately be combined, say by addition, unless it is known or imposed in advance that each equation is satisfied by the same set of values for the dependent variables. Accordingly, such combinations are permissible only in the process of solving simultaneous equations.

The usual combination is between a conditional equation and an identity or an arithmetical equation. For example, if we wish to solve the equation $3x + 2 = 6 - x$ we can write $x - 2 \equiv x - 2$ as an identity. Then, if we add, member by member, we obtain as a result that $4x = 4$. The solution is completed by writing $4 = 4$ as an arithmetical equation and dividing, member by member, the result being that $x = 1$.

The simple example above was given to demonstrate that the result of any combination between a conditional equation and an identity is another conditional equation. The number of solutions in the derived equation may be more than, less than, or the same as the number of solutions in the original equation. A necessary part of the technique of solving is in keeping track of these changes, if any.

A similar combination of two identities will

produce another identity. Likewise, most operations upon a single identity will yield another identity. For example, an identity for $\cos 2x$ can be derived by differentiating the identity for $\sin 2x$. In manipulations of this type the number of independent variables may be increased or decreased at will.

The meaning and properties of identities are most often violated in the matter of definitions. If, in any definition, a measurable quantity has its value made dependent upon the values of other quantities, then the defined quantity is a dependent variable. Therefore, the equation defining this dependence can never correctly be written as an identity. The most frequent deviation from this fundamental principle is in the definition of a function as "identically" zero in a region or an interval. The confusion is between the terms "always" and "identically". For example, on the x-axis y is always zero, yet the defining equation of the x-axis is never correctly written as an identity.

The above discussion does not imply that a definition can never be written with an identity as a necessary part of it. The definition of a homogeneous function given in the next section is one such example.

C. Homogeneous Functions.

Many of the functions used in the mathematical

analysis of economics are homogeneous in one or more sets of variables. In this section we will investigate some important properties of such functions.

Definition 1:3:6. Homogeneous Functions.*

> If $f(tx,ty) = t^n f(x,y)$ is an identity in x, y, and t for some value of n, then $f(x,y)$ is homogeneous of degree n in x, y, and t.

Actually, two things are defined above, the quality of being homogeneous and the order of the homogeneity. The quality is not measurable and requires the identity relationship in its definition. The order is measurable and the existence of the identical relationship is conditional upon the use of the proper value of n. Therefore, the above definition is partly identical and partly conditional.

Although the definition is stated for a function of only two variables, it is easily extended to a function of any finite number of variables. In most of the succeeding developments of this chapter the extension to any finite number of variables follows immediately.

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*Some of the theorems of this section have been suggested by G. Tintner, "Homogeneous Systems ...", Econometrica 16: 273-294 (1948). The proofs given here are much shorter and depend very little on the properties of determinants. In connection with Theorem 1:3:3, see C. G. Phipps, "A Note on Tintner's 'Homogeneous Functions'," Econometrica 18: 63-64 (1950).

Also, the concept can be extended to the case in which the function is homogeneous in only part of the variables present.

Definition 1:3:7. Partial Homogeneity.

If $f(x_1,\ldots,x_n,ta_1,\ldots,ta_m) = t^k f(x_1,\ldots,x_n,a_1,\ldots,a_m)$ is an identity in the x_i, a_j, and t for some value of k, then f is a homogeneous function of degree k in the a_j.

We now derive Euler's Theorem from Definition 1:3:6 in two simple steps. First, with the value of n fixed at the required number, we differentiate with respect to t, and second, since the derived equation is an identity in t, the variable t can be assigned the value one. We then have

(1:3:5) $$x\, f_x(x,y) + y\, f_y(x,y) = n\, f(x,y).$$

This is a mathematical statement of Euler's Theorem for two variables. The equation can be extended, by the same two steps, from the Definition 1:3:7.

For the proper values of n or k, these equations are identities in the remaining variables present. However, they cannot be written with the identity sign as long as either n or k appears as a literal quantity in the equation.

Being identities with the numerical values of n or k inserted, neither equation can be used as one of a set of simultaneous equations for the determination of any of the variables or parameters involved. This fact has a critical bearing upon later work where certain conditional equations appear in almost identical forms.

In order to interpret Euler's Theorem geometrically we consider the function $z = f(x,y)$ as any surface in 3-space. At a point (x_o,y_o,z_o) on the surface, the the equation of the tangent plane is

(1:3:6) $\quad (x-x_o)f_x(x_o,y_o) + (y-y_o)f_y(x_o,y_o) = z-z_o.$

It is at once evident that this tangent plane will pass through the origin if, and only if,

(1:3:7) $\quad x_o f_x(x_o,y_o) + y_o f_y(x_o,y_o) = z_o = f(x_o,y_o).$

Equation (1:3:7) is therefore a criterion of whether a plane tangent at a point on the surface passes through the origin. If it does, the equation of this tangent plane becomes

(1:3:8) $\qquad x\, f_x(x_o,y_o) + y\, f_y(x_o,y_o) = z.$

We now consider a third equation

(1:3:9) $\qquad x\, f_x(x,y) + y\, f_y(x,y) = z.$

This equation represents another surface in space. All points on the first surface, $z = f(x,y)$, which satisfy (1:3:7) will also lie on the second surface defined by (1:3:9). Therefore, the intersection of the two surfaces

is the locus of all points on the first surface at which the tangent plane passes through the origin.

If in (1:3:5) the value of $n = 1$ we have Euler's Theorem for a linear homogeneous function, or

(1:3:10) $$x f_x(x,y) + y f_y(x,y) \equiv f(x,y).$$

If $z = f(x,y)$ is the equation of a surface, where $f(x,y)$ satisfies (1:3:10), then every point (x,y,z) on this surface will satisfy (1:3:7). Consequently, we may state

Theorem 1:3:1. The tangent plane at every point on a surface represented by a linear homogeneous function passes through the origin.

The above theorem explains the relationship between Euler's Theorem and the passage of the tangent plane through the origin. In Part II we shall have occasion to note the confusion that has been engendered in the theory of marginal productivity by this tenuous connection.

While the four equations (1:3:7), (1:3:8), (1:3:9) and (1:3:10) have the same general form, the arguments which enter the functions are slightly different. Also, the first is an arithmetical equation, the second and third are conditional equations, and the fourth is an identical equation.

The general equation for a straight line in the xy-plane is $Ax + By = C$, where x, y are variable coordinates and A, B, C are parameters. This equation

is homogeneous of the first degree in the parameters and it represents all straight lines in the coordinate plane. If the equation is changed into the form $y = mx + b$ by dividing by B and solving for y, the new form contains only two parameters. However, this new form cannot be used to represent the family of lines perpendicular to the x-axis. Likewise, the form obtained by dividing by A cannot represent the family of lines perpendicular to the y-axis. And finally, the form obtained by dividing by C cannot be used to represent the family of lines which pass through the origin. This example illustrates the fact that it is sometimes necessary to use more parameters than the minimum number in order to represent all members of a geometric family by one equation. In cases where more than this minimum is used, both members of this type of equation are likely to be homogeneous of some common degree in the expanded set of parameters.

For the remainder of this section we shall take $f(x_1,\ldots,x_n,a_1,\ldots,a_m)$ as a function homogeneous of degree k in the m parameters a_j, and assume that the first and second partial derivatives of f with respect to each x_i, and that the first partial derivatives with respect to each a_j exist and are continuous in the domain under consideration.

In order to condense the notation we shall use a

single letter in place of a sequence of similar quantities. Thus,

$$x = (x_1, x_2, \ldots, x_n),$$

$$a = (a_1, a_2, \ldots, a_m),$$

$$ta = (ta_1, ta_2, \ldots, ta_m),$$

$$b = \frac{a}{a_1} = (1, \frac{a_2}{a_1}, \ldots, \frac{a_m}{a_1}), \qquad a_1 \neq 0.$$

We note that it is always possible to define the set b in this manner since the homogeneity will have no meaning when all of the parameters are zero, and the elements of the set of a's can be numbered so that the first one is different from zero.

Several conclusions can be drawn almost immediately from the above definitions. Noting that in the condensed notation the original function becomes $f(x,a)$, we state these conclusions as theorems.

Theorem 1:3:2. Any $f(x,b)$ is homogeneous of degree zero with respect to the a_j.

This is easily proved, since if ta_j replaces a_j in the expressions for b, the values of the b's, and consequently the value of the function, remains unchanged. In other words, the value of the function is multiplied by t to the zero power.

Theorem 1:3:3. If $f(x)$ does not contain the a_j or is constant, it may, as a special case, be regarded as homogeneous of zero degree with respect to the a_j.

Certainly the multiplication of all of the a_j by t will not change the value of the function. The converse is not true, however; the function may contain the a_j as parameters or not be constant for all values of x or both, and yet may be homogeneous of zero degree in the a_j.

Theorem 1:3:4. If $f(x,a)$ and $g(x,a)$ are two functions homogeneous in a of degree s and r respectively, then their product is homogeneous of degree $s + r$.

The above theorem follows immediately from the definition of homogeneity.

We now consider a change in the set of parameters in a function homogeneous of degree zero in that set. Let $f(x,ta) = f(x,a)$, where

$$a_j = g_j(c), \quad c = (c_1,\ldots,c_r), \quad j = 1,\ldots,m,$$

and where the g_j are all homogeneous of the same degree p in c; i.e., $g_j(tc) = t^p g_j(c)$.

If t is given the value $\frac{1}{a_1}$, we have

$$f(x,ta) = f(x,b) = f(x,a).$$

Let $$f(x,b) = f(x,\frac{a}{a_1}) = F(x,c),$$

and we have

$$F(x,tc) = f(x,\frac{t^p a}{t^p a_1}) = f(x,\frac{a}{a_1}) = F(x,c).$$

Therefore $F(x,c)$ is homogeneous of zero degree in c.

We may now state

Theorem 1:3:5. A sufficient condition that a homogeneous function of zero degree will remain of zero degree under a change of parameter is that the elements of the first set be homogeneous all of the same degree in the elements of the second set.

We consider now simultaneous equations involving homogeneous functions. Let the set of equations be

$$\begin{aligned} &Ax_1 + Bx_2 = C \\ &G_s(x_1,\ldots,x_n) = 0, \quad s = 1,2,\ldots,n-1, \end{aligned} \tag{1:3:12}$$

where the G_s do not contain the parameters A, B, C. These n equations contain the n unknowns $x_1,\ldots,x_n$, and various parameters. In general, these equations can be solved for the unknowns in terms of the parameters. The solutions can be indicated by

$$x_h = f_h(A,B,C,D,E,\ldots), \quad h = 1,\ldots,n,$$

where D,E,... represents the parameters which enter the solutions through the G_s.

Since $tAx_1 + tBx_2 = tC$ together with the $G_s = 0$ will lead to the same solution, we have that

$$f_h(tA,tB,tC,D,E,\ldots) = f_h(A,B,C,D,E,\ldots),$$

and the f-functions are shown to be homogeneous of zero degree in the set of parameters A, B, C.

We may now state

Theorem 1:3:6. If the equations in a set of simultaneous equations either do not contain any element of a set of parameters or are homogeneous of any degree in them, then the solution of this set of equations is homogeneous of degree zero in this set of parameters; the contrapositive also holds.

The conclusions of this theorem will play an important part in the economic applications.

The two following theorems pertain to derivatives of homogeneous functions. Let $f(x,a)$ be a function homogeneous in a and with the properties described above. By taking $t = \frac{1}{a_1}$, we deduce that

$$f(x,a) = a_1^k f(x,b).$$

If $\frac{\partial}{\partial x} f(x,a)$ is indicated by $f_x(x,a)$ and denotes the partial of f with respect to any x_1, we have at once

Theorem 1:3:7. If a function is homogeneous of degree k in one set of variables, and if the partial derivative is taken with respect to an element of a second set, the derivative is still homogeneous of degree k in the first set.

Likewise, if $a \neq a_1$, we have

$$f_a(x,a) = a_1^k f_a(x,b)\frac{1}{a_1} = a_1^{k-1} f_a(a,b).$$

If $a = a_1$, we may write the b in terms of some other element $a_j \neq 0$, in which case a similar equation holds. But if all the other a's are zero we have

$$f(x,a) = a_1^k f(x,1) \quad \text{and} \quad f_a(x,a) = k a_1^{k-1} f_a(x,1).$$

In every case we have

Theorem 1:3:8. If a function is homogeneous of degree k in a set of variables, and if the partial derivative is taken with respect to an element of this set, then the derivative is homogeneous of degree $k - 1$ in this set.

The final theorem of this section is concerned with maximum and minimum values of homogeneous functions.

Consider the set of $N + 1$ functions

$$g(x,a) \quad \text{and} \quad h_r(x,a), \quad r = 1,2,\ldots,N < n,$$

where continuous partial derivatives are assumed as above. Also, each of the $N + 1$ functions is assumed to be independent of the a or homogeneous of some degree in the a. We wish to investigate points of maximum or minimum values of g subject to the N constraints of $h_r(x,a) = 0$.

We begin by transforming the problem into a similar

one with the same solutions for the x_i. The $h_r(x,a) = 0$ may be changed into $h_r(x,b) = 0$ by dividing each equation through by the proper power of a_1 ($\neq 0$). Also, $g(x,a) = a_1^k g(x,b)$. Hence, if $a_1 > 0$ or if k is even, then the two g-functions will attain their maximum and minimum values together. If $a_1 < 0$ and if k is odd, then the maximum of one will correspond with the minimum of the other.

The points of maximum and minimum value of the new problem can be found and tested in the usual way. The solutions for this problem, being solutions of a set involving homogeneous equations, will, by Theorem 1:3:6, yield the values of the x_i in terms of the b_j. Hence we may state

Theorem 1:3:9. If a function g which is to be maximized or minimized and the restrictive conditions $h_r = 0$ are either independent of the a_j or homogeneous of any degree in them, then the coordinates of the maximum or minimum are homogeneous of zero degree in the a_j.

PART II

APPLICATIONS TO ECONOMICS

CHAPTER 1

PRODUCTION IN A STATIC ECONOMY

The theory of production in a static economy is based primarily upon three independent conditions. Many subsidiary assumptions vary the particular form which the principal theory may take.

The first of these conditions is an assumption which deals with the production of one or more finished articles from certain factors, sometimes tabulated as land, labor, capital and entrepreneurship. The behavior of the producer is presumed to be such that he does the best that he can and makes the most out of what he has.

In the static economy it is assumed that the given technical knowledge remains constant and is available to all. On this basis, two producers will, when given the same amounts of the basic factors, use them in the same, or in similar, ways to achieve identical amounts of production. As a consequence, output becomes a mathematically definable function of input.

The static economy is in contrast with a dynamic economy in which technical knowledge may change, new materials may become available, restrictions may be imposed upon the dissemination of information regarding production, etc. Any factor of this nature will alter

the functional relationship between output and input.

The second condition amounts to a definition of total costs. The costs are composed of the fixed expenses, if any, plus the costs of the basic factors used. This is entirely an economic matter and in no way depends upon how these factors are used or upon the quantity of finished commodities produced.

It should be noted that the functional relationship between output and input is determined by technical considerations alone. This function, expressed in the form of an equation, will be known as a technical equation, while the equation resulting from the second condition will be known as an economic equation.

The third condition is the existence of free competition. This condition results in the fact that, in the long run, the prices at which the finished commodities are sold become equal to their minimum average costs of production. Quite often this conclusion is taken as an assumption but it appears to be more logical to establish the conclusion as a theorem based upon the existence of free competition and the economic behavior of the individual.*

In the short run, or if free competition does not exist, the market price may exceed the cost of production.

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*This is done on p. 86, following the development of the minimum average cost of production.

It is then possible for the producer to make a profit, and we assume that each producer will maximize this profit.

The maximum profit will be limited by both the cost of the basic factors and the technical restrictions upon production. One of the equations used in determining this profit is obtained by the application of theorems of Part I; and this equation, since its terms depend upon both technical and economic matters, will be of a mixed type, not wholly technical nor wholly economic. An equation, similarly derived, and which aids in the determination of the minimum average cost of production, is likewise one of this mixed type.

It is important, for purposes of interpretation, to keep in mind the type of equation whose meaning is being interpreted. One famous equation, developed later, depends only on technical matters and the existence of free competition. Identified in this manner, its interpretation becomes clear, although every author who has dealt with it has given it an erroneous interpretation.*

A. The Essentials of Marginal Production.

The theory of marginal production was first formulated at the close of the last century. This valuable contribution to economic theory was largely couched in words, since the majority of the authors and readers of economics at that time were relatively untrained in mathematics. An

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*See section B of this chapter, p. 73 ff.

examination of the principal references concerning the theory seems to indicate that conclusions were deduced first and such mathematics as was convenient used to substantiate the conclusions. As a result, errors and misconceptions were introduced, both in the mathematical manipulations and in the interpretations. Many subsequent writers on the subject followed in the missteps of the earlier authors. A search of the literature has failed to reveal an exposition of the subject which is both complete and correct from a mathematical point of view.

The sources of the confusion which exists on the subject can be summed up under three main headings.

(i) There are several forms of the theory and of the problems involving the theory, each with a different set of underlying assumptions. Sometimes these assumptions are not stated at all, and seldom are they clearly set forth. The fact that other assumptions would lead to other conclusions is frequently ignored. Sometimes a conclusion derived under one form of the theory holds true under another form, sometimes not.

(ii) Often there is no distinction between what in mathematics are termed necessary and what are termed sufficient conditions. Also, no general effort appears to have been made to distinguish between necessity on mathematical grounds and necessity on economic grounds.

When these distinctions are made many seeming contradictions either disappear or are easily explained.

(iii) The manipulation and interpretation of the mathematics is often faulty. In some cases, the errors either invalidate the conclusions or throw them open to serious question. In other cases, the mathematical treatment is far from complete and adds nothing to the argument. Or to phrase it more strongly, in some cases authors have attempted to add the prestige which is afforded by a mathematical proof to the weight of their conclusions without strictly adhering to the rules of mathematics.

The primary purpose of this section is to establish a consistent mathematical basis for the various forms of the theory of marginal production and to interpret the results. In so doing, it will be necessary to traverse ground that has been covered extensively in other published material. However, it is the intention to point out certain pitfalls along the way, and after pointing them out, to avoid them. Also, from the mathematical point of view, many results and conditions will change radically in importance. In addition, new interpretations will be developed and new simplifications, only hinted at in the present literature, will be effected.

The number of variables in much of the treatment will be intentionally restricted in order that a picture

in two and three dimensions may accompany the analysis. The analysis, however, will be completely general and can, without further restrictions, be extended to any number of variables.

We begin with the simple case in which the production of a finished commodity involves only two basic factors, X and Y. We designate by x and y the number of units respectively of the basic factors used in the production of z commodities. We have that

(2:1:1) $$z = f(x,y)$$

is the production function, or equation of production, certain of whose properties are to be determined.

We denote by p_x and p_y the respective unit prices of the factors. Then, if C represents the total cost, we have

(2:1:2) $$C = x\,p_x + y\,p_y .$$

Several economic assumptions must be made in order that the above production and cost functions have the continuity and uniqueness necessary for our mathematical manipulation.

The factors of production must be infinitely divisible, so that x and y may be continuous variables. This is the divisibility assumption.

It must be assumed that all combinations of x and y can be used in the production of the finished commodity; i.e., that z is defined for all pairs of

values of x and y. This is the substitutability assumption which is usually stated in the form that the same number of finished commodities can be produced by using less X (or Y) and correspondingly more Y (or X) in its place.

In order to provide uniqueness for C, it is necessary to assume that all factors acquired can be, and are, used.

In order to provide uniqueness for z it is necessary to assume that the producer always uses the factors which he has acquired to construct the maximum number of commodities which his technical knowledge permits. The static qualification upon the economy implies that this technical knowledge does not change, and hence that the production function remains fixed in form.

Thus, we have a given, continuous production function. Later in the chapter we will consider certain modifications of this function.

The cost function is an obvious and explicit function, while the production function is indefinite and implicit. How the latter is determined is an economic matter and for the present we assume it to be given.

To derive the theory in its simplest form we assume that the unit prices, p_x and p_y, are constant, regardless of the quantities that may be used in the production.

We will first consider the relationship between cost and production. Figure 2:1:1 is a picture of the first quadrant of xy-plane. The third dimension is along the z-axis, perpendicular to the page at O. The straight lines and the curve of this figure will be explained in the following development.

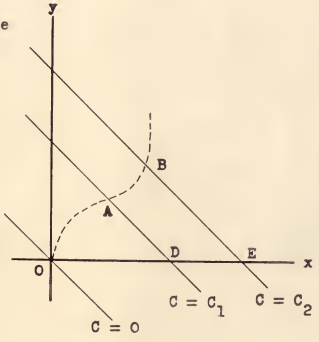

Figure 2:1:1.

The straight lines AD and BE are lines of constant total cost, or more briefly, cost lines. These lines are parallel because the prices, p_x and p_y, are constant.

The equation of AD is $x\,p_x + y\,p_y = C_1$, where C_1 is some fixed value. Similarly, the equation of BE is $x\,p_x + y\,p_y = C_2$, where C_2 is another fixed value. The line passing through the origin is the special case where $x\,p_x + y\,p_y = 0$. If anywhere along the line AD a point is chosen whose coordinates are x and y, then the cost of x units of the X-factor and y units of the Y-factor is C_1.

In the xy-plane, each cost equation represents a straight line. In space, the equations are the same but each represents a plane through its respective line and perpendicular to the xy-plane.

We now consider the production surface, $z = f(x,y)$, which lies above the first quadrant of the xy-plane. Each cost plane intersects this surface in a curve. The production may vary along each curve but the cost at every point remains the same. We designate this family of curves as "isocost curves."

We now assume that for each given cost there is a point at which the production is greater than at any other along the same isocost curve. We also assume that each producer operating at a given cost adjusts his methods of production so as to attain this maximum output. This is an extremely important assumption, since without it there can be no mathematical theory of production. This condition of producing the greatest possible amount for a given cost will be designated by the phrase, "operating at maximum efficiency."

Let Q denote the quantity produced by operating at maximum efficiency. For each C there will exist a Q, hence Q can be expressed as a function of C, and which we write as

(2:1:3) $$Q = F(C).$$

The above function can be pictured as a series of points on the production surface. In general, the points will all lie along a smooth curve which corresponds somewhat, although not exactly, to the continental divide on maps. These curves are sometimes given the name "expansion paths." The projection upon the xy-plane of one such curve is the dotted line OAB of Figure 2:1:1.

As far as the economic aspects of the problem are concerned, Q may be expected to increase with an increase in C, although not always at the same rate. Mathematically, this means that $\frac{dQ}{dC}$ is never negative. In the work which follows it will sometimes be more convenient to use the reciprocal of the above derivative and we note that it will be non-negative also.

It is possible to develop the entire problem of production using the function $Q = F(C)$. However, it is more instructive to investigate the properties of the production surface by studying the entire surface. It must be remembered, though, that we are assuming production at maximum efficiency, which requires that all production take place along the curve $Q = F(C)$.

We will now determine a condition on the production surface along the curve of maximum efficiency and examine some of the properties of the tangent plane.

From (2:1:1) and (2:1:2) we have

(2:1:4) $$dz = f_x dx + f_y dy, \quad \text{and}$$

(2:1:5) $$dC = p_x dx + p_y dy,$$

where f_x and f_y are the usual partial derivatives of $f(x,y)$ with respect to x and y respectively. However, p_x and p_y do not denote partials, but unit prices. If the cost is held constant, then $dC = 0$; and on the isocost curve, where $z = Q$, and is a maximum, then $dz = 0$ also. However, along the isocost curve dx and dy are not zero, and along the cost line they have the same value as at the corresponding point on the curve. Thus, we obtain

(2:1:6) $$\frac{p_x}{f_x} = \frac{p_y}{f_y} = \lambda .$$

The quantity f_x is the physical marginal productivity of the X-factor. It can be regarded as the increase in production made possible by using one more unit of the X-factor when there is already being used the amounts x and y of the factors.

By solving (2:1:1), (2:1:2) and (2:1:6) simultaneously for x, y, and z, we obtain the coordinates of the point on the given cost line where the maximum production, z, is attained. If there are more than two variables entering the production function and the cost equation, the proportionality expressed in (2:1:6) holds and extends similarly to all of them.

Before examining the plane tangent to the production

surface, it may be helpful to review the geometry involved. If a point moves from a position (x,y) on a plane curve to a second position indicated by the coordinates $(x + dx, y + dy)$, the second position is always on the line tangent to the curve at the first position. Likewise, the point in space $(x + dx, y + dy, z + dz)$ always lies in the plane tangent to the surface at the point (x,y,z). To indicate the second position on the surface we use the notation $(x + \Delta x, y + \Delta y, z + \Delta z)$. It is true that $dx = \Delta x$ and $dy = \Delta y$ when x and y are the independent variables, but a similar equality for dz and Δz does not hold when z is the dependent variable. Confusion on this point has led some authors in the field of econometrics to assign properties to the production surface that are possessed only by the tangent plane.

If a point moves in the tangent plane from (x,y,z) to $(x + dx, y + dy, z + dz)$, it moves a definite amount in a definite direction. Of the two, the direction is the more important here, and it is determined by the direction cosines, which, in turn, depend upon dx, dy and dz.

We now form the ratio $\frac{dC}{dz}$ from (2:1:4) and (2:1:5). For points along the curve of maximum efficiency this ratio reduces, with the aid of (2:1:6), to λ. Since this quotient does not contain dx or dy, it is independent of their values. Geometrically interpreted, this leads

to the first property of the tangent plane, or

Theorem 2:1:1. The ratio $\frac{dC}{dz}$ is the same in all directions in the plane tangent to the production surface at any point along the curve of maximum efficiency.

Along the curve of maximum efficiency $z = Q$, and hence $dz = dQ$. Therefore, along this curve we have

$$\frac{dC}{dQ} = \frac{p_x}{f_x} = \frac{p_y}{f_y} = \lambda \quad . \tag{2:1:7}$$

Thus λ is found to be equal to $\frac{dC}{dQ}$, a quantity which is defined as the marginal cost of production. This quantity is approximately equal to the cost of producing just one more article if the producer is already operating at maximum efficiency.

We now state a theorem which is the translation of equations (2:1:6) and (2:1:7) into economic language.

Theorem 2:1:2. If a producer is operating at maximum efficiency, then the per unit price paid for each factor entering into the production is proportional to that factor's physical marginal productivity, and the proportionality element is equal to the marginal cost of production.

It should be noted that the above conclusions were reached with no reference to the price at which the finished article is sold. They are also independent of the amount

which is produced. These conditions will be fulfilled whether the producer makes a profit, breaks even, or sustains a loss. They result solely from the fact that the greatest possible amounts of commodities are produced for a given cost.

We now examine a second property of the tangent plane. Let $x\,p_x + y\,p_y = C_1$ be one of the cost lines. By setting $z = 0$, the plane $x\,p_x + y\,p_y - \lambda z = C_1$ is found to contain this cost line. The value of λ is taken as the value of $\frac{p_x}{f_x}$ at the point of tangency. The direction cosines of the normal to this plane are proportional to the coefficients p_x, p_y, and $-\lambda$. The direction cosines of the normal to the tangent plane are proportional to the quantities f_x, f_y, and -1. By (2:1:6), these latter three quantities are proportional to the three coefficients. Therefore the two normals have the same direction and the two planes are parallel. Thus a line lying in one of the planes (the cost line) is parallel to the other plane (the tangent plane), and we may state

Theorem 2:1:3. A plane tangent to the production surface at any point along the curve of maximum efficiency is parallel to each line in the family of cost lines.

Thus far we have not considered the average cost of production. It is probable that different quantities of

production will yield different average costs. It is equally likely that there is at least one quantity of production for which the average cost is a minimum. We will now locate this minimum average cost of production.

Designating the average cost by the letter p without a subscript, we have

$$p = \frac{C}{z} = \frac{x\,p_x + y\,p_y}{f(x,y)} \,. \tag{2:1:8}$$

The above equation can be handled more readily if it is written in the form

$$p\,f(x,y) = x\,p_x + y\,p_y \,. \tag{2:1:9}$$

Noting that the derivatives of the p's are zero (p_x and p_y being constants and p being a minimum), we differentiate (2:1:9) partially, once with respect to x and once with respect to y, obtaining

$$\begin{aligned} p\,f_x &= p_x, \quad \text{and} \\ p\,f_y &= p_y. \end{aligned} \tag{2:1:10}$$

By eliminating p from the above equations we obtain the same relation as in (2:1:6). This implies what we should have known all the time; that is, the minimum average cost is attained only by producing at maximum efficiency.

The point at which the minimum average cost of production is attained can be found by solving (2:1:9) and (2:1:10) simultaneously for x, y and p. The cost and quantity of production can then be determined from

(2:1:1) and (2:1:2). Of all possible solutions to these equations, we denote by (x_o, y_o, z_o) the one with the smallest value for p.

We denote the minimum average cost of production, if such a minimum exists, by p_o, and since it is equal to $\frac{p_x}{f_x}$ by (2:1:10), this average is also equal to the marginal cost of production there.

If the values of p_x and p_y from (2:1:10) are substituted in (2:1:9), and the common factor p divided out, we have

(2:1:11) $\quad f(x_o, y_o) = x_o f_x(x_o, y_o) + y_o f_y(x_o, y_o)$.

The above relation holds at the point of minimum average cost. It also holds at every other point, maximum or minimum, which is a solution of equations (2:1:9) and (2:1:10). However, these latter points offer no special concern.

At this point it might be stated that equation (2:1:11), in one form or another, is perhaps the most misunderstood equation in the theory of marginal production. A detailed analysis of this topic will follow in a later section.

The equation of the tangent plane at (x_o, y_o, z_o) is

(2:1:12) $\quad z - f(x_o, y_o) = (x-x_o) f_x(x_o, y_o) + (y-y_o) f_y(x_o, y_o)$.

With the aid of (2:1:11) the above equation reduces immediately to

(2:1:13) $$z = x\, f_x(x_o, y_o) + y\, f_y(x_o, y_o).$$

If both sides of this equation are multiplied by p_o, and if (2:1:10) is used to replace $p_o f_x$ by p_x and $p_o f_y$ by p_y, then we have

(2:1:14) $$x\, p_x + y\, p_y - z\, p_o = 0.$$

The above equation enables us to state the two following theorems:

Theorem 2:1:4. If a plane is tangent to the production surface at a point where the average cost is a minimum, then this plane passes through the origin along the line $x\, p_x + y\, p_y = 0$.

Theorem 2:1:5. If the smallest value of p has been chosen for p_o, then no point on the production surface is above the tangent plane.

In the preceding theorem the term "above" is used to mean that side of the production surface opposite the xy-plane. By using the value of C from (2:1:2), equation (2:1:14) can be written in the form $p_o = \frac{C}{z}$. Any point above the tangent plane would have a larger z, for the same x and y, than the corresponding z on the plane. Since z would be larger, $\frac{C}{z}$ would be smaller. Consequently, at this point above the tangent plane the average cost of production would be smaller than p_o.

Since this is a contradiction, no point on the production surface lies above the plane tangent at the point where the average cost of production is a minimum.

To obtain the geometric picture we visualize a plane passing through the line $x\,p_x + y\,p_y = 0$ and the z-axis. We rotate this plane about the line as an axis toward the first quadrant of the xy-plane. The first point of contact between the rotating plane and the production surface is the point (x_o, y_o, z_o) at which the average cost is p_o.

If our problem has a solution, the production surface must have a shape such that contact by tangency (not by intersection) is possible. We classify the situations under which this can occur as follows:

(i) The plane and the surface may have only one point, or several isolated points, of contact. If there is a single point, the profile of the curve $Q = F(C)$ will have a single "hump." Any operator, wishing to produce at minimum average cost, must operate at the scale indicated by the point of tangency. If there is more than one such point, the producer is free to choose between them.

(ii) The plane and the surface may have in common a line, straight or curved, finite or infinite in length. In this case, the operator can vary his production and is limited only by the extent of the line. Here the producer can increase or decrease production in each "factory". This is in contrast to the first case where production

can be increased only by building new and separate "factories."

(iii) This is a special case of (ii) where the line is a straight line through the origin. In this case the proportions of the basic factors remain constant for all amounts of production. In particular, a production function which is defined by a linear homogeneous equation will have this property for all ratios of the two prices, p_x and p_y. In fact, this very special case is the only type of surface which will yield a straight line for all possible ratios, although many other surfaces will yield a straight line of contact for certain ratios.

B. Euler's Theorem and the Theory of Marginal Productivity.

Euler's theorem on homogeneous functions has been associated with the theory of marginal productivity from the time that the theory was first put in mathematical form. Most of the early authors used this theorem as an integral part of their development. Many modern writers ignore any reference to it, and the few that have attempted to relate it to marginal productivity have done so with certain reservations. Some are beginning to doubt its applicability, yet many will be surprised to find that the Euler theorem has no connection with marginal productivity.

The difficulties seem to stem from the fact that three forms of an equation are very nearly alike algebraically but have entirely different geometric interpre-

tations. The first two forms are the equations (2:1:11) and (2:1:13) of this chapter.

The equation (2:1:11) may be written with z_o replacing $f(x_o,y_o)$. This equation contains no variable quantities. It is satisfied, as we have shown, by the coordinates of any point on the production surface at which the tangent plane passes through the origin. As far as the economic application is concerned, the point at which the average cost of production is a minimum is one such point.

The equation (2:1:13) is the equation of the plane tangent to the production surface at the point whose coordinates satisfy (2:1:11). It is immediately obvious that this plane passes through the points (x_o,y_o,z_o) and $(0,0,0)$.

There are infinitely many surfaces with points at which the tangent plane has the above property. Many of these may serve as a production surface on which there is a point of minimum average cost. Conversely, any surface not having a point with this property cannot be a production surface with a point of minimum average cost.

We now consider a surface defined by $z = f(x,y)$, where f is homogeneous of degree one. Such a surface is a "ruled" surface, or one on which a straight line can be drawn on the surface through any point, with all of the rulings passing through the origin. The tangent

plane at a point on a ruled surface always contains the ruling, or rulings, through that point. Therefore, every point on a surface represented by a linear homogeneous function has this property.

We have here derived Euler's theorem as it applies to a homogeneous production function, rather than the converse. Our development shows that such a surface has properties which insure a minimum average cost of production but that it is a very special case and is by no means necessary. This is one example of the confusion between necessary and sufficient conditions.

We note here that the linear homogeneous production function presents an additional complication; that is, that the minimum average cost of production point is not unique. The locus of such points is a straight line passing through the origin. Since C and z are both linear homogeneous functions of x and y, their quotient, $\frac{C}{z}$, is unchanged when x and y are replaced by λx and λy. With such a production surface, the producer may choose any scale of operation, but will always use his factors of production in the same proportion.

We now examine the behavior of the tangent plane of (2:1:13) as the ratio of p_x to p_y varies from zero to infinity. This variation will cause the line $x\,p_x + y\,p_y = 0$ to rotate in the xy-plane from coincidence with the x-axis ($p_x = 0$) to coincidence with the y-axis ($p_y = 0$).

Since the tangent plane passes through this line and rests upon the production surface, the point (assumed unique) of contact between this plane and the surface will, in general, move over the production surface as the tangent plane rolls from side to side. The locus of this point of contact is a locus upon the production surface of points at which the tangent plane passes through the origin. It will be the locus of all such points only if there is a single solution to the equations (2:1:9) and (2:1:10), or if the multiple solutions all result in the same p_o.

The analytic formulation of this locus involves a third equation having a form similar to (2:1:11) and (2:1:13). It is obtained by eliminating p, p_x, and p_y from (2:1:9) and (2:1:10), or

(2:1:15) $$z = x\, f_x(x,y) + y\, f_y(x,y).$$

Although the above equation has a form similar to the previous equations, it has an entirely different interpretation. The surface represented by this equation will not be a plane unless the partial derivatives are identically constant. This can occur only when the production surface is a plane. But (2:1:1) and (2:1:6) would be inconsistent unless $f_x = p_x$ and $f_y = p_y$, in which case the tangent plane coincides with the production surface. Hence, except in cases where p_x and p_y are required to be held fixed, (2:1:15) represents

a curved surface.

Since (2:1:15) does not contain the prices, either implicitly or explicitly, it is independent of their ratio. However, since this equation is derived from (2:1:9) and (2:1:10), it is satisfied by all points of tangency which these latter equations define. These points of tangency also lie on the production surface. Therefore, the intersection of the surfaces defined by (2:1:1) and (2:1:15) determines the locus of all points on the production surface at which the tangent plane passes through the origin.

If each point of the locus in the preceding paragraph is joined to the origin by a straight line, a linear homogeneous surface will be generated. The outer portion of this surface fits over and encloses the production surface. Since the two surfaces are tangent at the points of the above locus (they may intersect at an angle at other places), one surface may serve as an approximation of the other in a neighborhood of their tangency.

In particular, this homogeneous surface will show a proportionate increase in production for a proportionate increase in all of the factors of production. Hence, a statistical test of production, for relatively small changes in the factors, might result in a surface which appears to be linearly homogeneous.

Mathematically, the linear homogeneous surface so constructed is the envelope of the family of planes of (2:1:13) as they roll over the production surface. The equation of this envelope can be determined from the given conditions. This envelope has a number of the properties usually, and erroneously, ascribed to the production surface.

To illustrate the foregoing argument we will set up a simple example of a specific production surface and its accompanying equations.

Let the production surface be defined by

(2:1:1-a) $$z = \sqrt{100 - (x-10)^2 - (y-10)^2}$$

when x and y are within the circle $(x-10)^2 + (y-10)^2 = 100$, and let $z = 0$ elsewhere.

Take the ratios of the prices as unity, or $p_x = p_y$.

We now have

(2:1:9-a) $$p\,z = x\,p_x + y\,p_y, \quad \text{and}$$

(2:1:10-a) $$p\,\frac{\partial z}{\partial x} = p_x,$$

$$p\,\frac{\partial z}{\partial y} = p_y.$$

Solving, we have

$$p_x = p_y, \qquad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y},$$

$$z = x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = \frac{x(10-x)}{z} + \frac{y(10-y)}{z}, \quad \text{and}$$

$$z^2 = 2x(10-x) = 100 - 2(10-x)^2;$$

whence, $x = 5, \quad y = 5, \quad z = 5\sqrt{2}$.

Thus, for (x_o, y_o, z_o) we have $(5,5,5\sqrt{2})$.

Equation (2:1:11) becomes the arithmetic equation

(2:1:11-a) $$5\sqrt{2} = \frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}} .$$

Equation (2:1:13) becomes

(2:1:13-a) $$z = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} ,$$

or the equation of the plane tangent to the production surface at $(5,5,\sqrt{2})$ and passing through the origin. Equation (2:1:15) becomes

(2:1:15-a) $$z = \frac{x(10-x) + y(10-y)}{\sqrt{100-(x-10)^2-(y-10)^2}} ,$$

or a fourth degree equation independent of the price ratios but derived from (2:1:9-a) and (2:1:10-a), and satisfied by all points of tangency which these latter equations define.

The intersection of (2:1:1-a) and (2:1:15-a) determinesthe locus of all points on the production surface at which the tangent plane passes through the origin. In this example we may simplify (2:1:15-a) by substituting z for the denominator, thereby obtaining the equation $x + y = 10$. This is the equation of the projection on the xy-plane of the curve of intersection of (2:1:1-a) and (2:1:15-a).

For a different ratio of the prices the equations (2:1:11-a) and (2:1:13-a) will be changed but the equation

(2:1:15-a) will remain the same.

The envelope of the family of tangent planes is the cone $z^2 = 2xy$. This can be written in the linear homogeneous form $z = \sqrt{2xy}$.

When $x = 5$ and $y = 5$, we have that $z = 5\sqrt{2}$ from both the envelope and the production surface. However, when $x = 6$ and $y = 6$, we have that $z = 8.485...$ from the envelope and that $z = 8.246...$ from the production surface. Thus we see that in the neighborhood of points of tangency the envelope may serve to approximate the production surface but that the two surfaces, and their equations, are quite different.

C. Variations in the Basic Problem.

In this section we will investigate the effects of changes in the conditions which gave rise to the simple theory previously discussed.

The first variation which we shall consider is the introduction of certain fixed costs, such as a license fee, which are incurred before or during the production and which are independent of the amount produced.

The equation of production, (2:1:1), will remain unchanged, but the cost equation, (2:1:2), will be modified to

$$C = x\,p_x + y\,p_y + C_o, \tag{2:1:16}$$

where C_o is the total amount of the fixed costs.

The cost lines have been shifted, hence the curve of maximum efficiency becomes

(2:1:17) $$Q = F(C - C_o).$$

Since dC, dz and dQ remain unchanged, equations (2:1:6) and (2:1:7) are the same as before.

Equation (2:1:8) is changed only by the addition of C_o to the numerator of the rightmost fraction, hence equation (2:1:9) will be

(2:1:18) $$p\ f(x,y) = x\ p_x + y\ p_y + C_o.$$

The equations of (2:1:10) remain unchanged, but the simultaneous solution of (2:1:10) and (2:1:18) will produce a new set of values for x_o, y_o, p_o.

Similarly, equation (2:1:11) becomes

(2:1:19) $$f(x_o,y_o) = x_o f_x(x_o,y_o) + y_o f_y(x_o,y_o) + \frac{C_o}{p_o}.$$

The equation of the tangent plane as given by (2:1:12) becomes

(2:1:20) $$z = x\ f_x(x_o,y_o) + y\ f_y(x_o,y_o) + \frac{C_o}{p_o}.$$

Previously the tangent plane intersected the xy-plane in the line $x\ p_x + y\ p_y = 0$, and now it intersects in the line $x\ p_x + y\ p_y + C_o = 0$.

The form of (2:1:20) shows that there is no longer a question of a linear homogeneous production function, since the tangent plane at the point of minimum average

cost of production no longer passes through the origin. In fact, such a homogeneous function would yield no maximum when fixed costs exist.

The above discussion demonstrates the absurdity of the insistence that the production surface be linearly homogeneous. If there is a minimum average cost, the mere addition of a ten dollar license fee would mathematically prohibit linear homogeneity.

It is possible that the producer may purchase the basic factors of production in such quantities as to affect their prices. We will now investigate the results of such changes.

We will assume that a change in the price of one factor results solely from a change in the demand for that factor, and that it is unaffected by the demand for other factors. The quantity $\frac{dp_x}{dx}$ may be positive or negative, and it may be zero at a maximum or minimum without p_x being constant.

Under these conditions, dC becomes

(2:1:21) $$dC = x\left[\frac{dp_x}{dx} + p_x\right]dx + y\left[\frac{dp_y}{dy} + p_y\right]dy .$$

Thus the quantities in the above brackets replace p_x and p_y, respectively, in (2:1:6) and (2:1:10). Otherwise, no changes occur in notation. However, there will be certain changes in the results. The cost lines, for example, will become curved lines. Cylindrical

surfaces (rather than planes) through these curved lines, and with their elements perpendicular to the xy-plane, will intersect the production surface in isocost curves as before. For each C there will be a maximum Q, hence Q will be a function of C as indicated by (2:1:3), although the curve of maximum efficiency will probably lie in another place upon the production surface.

The point of minimum average cost of production can be determined by solving (2:1:9) simultaneously with the new forms of (2:1:10).

Since the variable prices are not affected by fixed costs, both fixed initial costs and variable prices of the basic factors may be introduced simultaneously into the problem without altering the general method of solution.

We will now consider the sale price of the product and profits. Designating the market price of a product by u, we find that there are three possible mathematical relations between u and the minimum average cost of production, p_o.

The first relation is that $u < p_o$. Under our conditions of a static economy, no producer would operate in this situation.

The second relation is that $u = p_o$. In this case, every producer must operate on exactly the same scale.

Each must purchase the amounts of the factors which yield the minimum average cost of production and use them to produce a certain amount of the finished commodity. This means that each producer must build his plant in an exact size, buy exact amounts of the basic factors, produce an exact amount of the finished commodity, etc.

The return from the sales is $p_o z_o$. The cost of this production is $x_o p_x + y_o p_y + C_o$ (where C_o may be zero or not), and we have

(2:1:22) $\qquad p_o z_o = x_o p_x + y_o p_y + C_o.$

But from (2:1:10) we have that $p_o f_x = p_x$ and $p_o f_y = p_y$, and we may transform (2:1:22) into

(2:1:23) $\qquad p_o z_o = x_o f_x p_o + y_o f_y p_o + C_o.$

According to many authors, the theory of marginal productivity reaches its climax with the facts implied in the above equations. In substance, the statement is, "If the basic goods and services are paid according to their marginal productivity, the total product is exactly exhausted." This statement does not correctly describe what actually takes place.

Consider the situation where "the total product is exactly exhausted." This situation can be attained at any point on the production surface. All that is required is that the market price be equal to the average cost of production at the given point. In other words, this

situation is attained whenever the producer "breaks even" on his production.

Consider the situation where "the basic goods and services are paid according to their marginal productivity." This condition is attained whenever the producer is operating along the curve of maximum efficiency. Whenever the producer is operating in this manner, the basic goods and services are paid according to their marginal productivity regardless of whether the producer is operating at a profit, without a profit, or at a loss. Even this statement is incorrect in its implications. The marginal productivity of a good does not fix its rate of pay as implied by the above statement. Cause and effect are reversed. It is the rate of pay which determines the marginal productivity. Presumably a good X is used in such quantity that its marginal productivity, f_x, has decreased and at the same time the marginal cost of production, λ, has increased until the ratio $\frac{f_x}{\lambda}$ equals p_x.

Therefore, these two independent conditions can be attained at any point along the curve of maximum efficiency if the market price is equal to the average cost of production at the given point.

Since the two conditions under consideration are independent, one cannot be made subordinate to the other; that is, the statement as it stands, and without modifi-

cation, is not true. The problem does not have a unique solution even if both statements are given coordinate rank and required to hold simultaneously. What, then, is implied by their juxtaposition?

The implication lies in the assumption of "free competition." Free competition brings the market price down to the minimum average cost of production. We can establish this fact with the following double syllogism:

(i) In a static economy, production will not take place unless the market price of the product is at least equal to the average cost of production.

(ii) The lowest average cost of production is p_o.

(iii) Therefore, there will be no production unless the market price is at least p_o.

(i) Free competition brings the market price to the lowest level at which there will be a supply.

(ii) The lowest level at which there will be a supply is p_o.

(iii) Therefore, free competition brings the market price down to the minimum average cost of production.

Under free competition, in the long run, there can be no profit. Since there is no profit, the product is exhausted; that is, the entire income from sales is expended for goods and services used in the production.

Since the operator is producing at maximum efficiency to avoid a loss, he is using these goods and services in amounts such that their prices are proportional to their marginal products. The proportionality factor is the marginal cost of production which, at this point, is equal to the average cost of production.

It is, therefore, the assumption of free competition which causes the product to be exhausted. This is an economic matter which has little to do with the technical matters of production. These technical matters merely establish a limit below which the price cannot go. On the other hand, the technical matters determine how the goods are used and in what quantities. The two causative factors are independent although they may both bear upon the same situation.

The third relation is that $u > p_o$. In this case, it is possible for the producer to make a profit. We write the profit equation as

(2:1:24) $$\pi = u\, f(x,y) - (x\, p_x + y\, p_y + C_o),$$

where π is the profit.

The point at which π is a maximum can be found by solving the above equation simultaneously with the equations obtained by setting the two partial derivatives of π equal to zero. This will yield the values of x, y and π. The corresponding values of z and C

can be found by substituting these values in (2:1:1) and (2:1:16). The point (x,y,z) will, of course, lie somewhere along the curve $Q = F(C)$, since only the most efficient operation will return the greatest profit.

Additional information can be obtained by writing the profit equation in a form which involves only the cost and quantity of production, or

(2:1:25) $$\pi = u\,Q - C\ .$$

Along the curve $Q = F(C)$ we take Q as the independent variable. Now, we wish to maximize π. Noting that u is constant, we have

(2:1:26) $$\frac{d\pi}{dQ} = u - \frac{dC}{dQ} = 0.$$

Thus, we have that $u = \frac{dC}{dQ}$, or that the market price is equal to the marginal cost of production.

We observe that (2:1:25), (2:1:26) and (2:1:27) all hold, even if the prices of the factors vary with the amounts used.

If the market price varies with production, we again use equation (2:1:25), taking Q as the single independent variable, since u is connected with Q by an equation not specified here. We differentiate with respect to Q and have

(2:1:27) $$\frac{d\pi}{dQ} = u + Q\,\frac{du}{dQ} - \frac{dC}{dQ}\ ,$$

If the profit is to be maximized, then the above equation will be equal to zero and we have

(2:1:28) $$\frac{dC}{dQ} = u + Q\,\frac{du}{dQ}\ .$$

If we denote the total revenue by R and define it by the equation $R = Q\,u$, we have

(2:1:29) $$\frac{dR}{dQ} = u + Q\,\frac{du}{dQ} = \frac{dC}{dQ}\ .$$

Hence the maximum profit occurs at a point where the marginal cost equals the marginal revenue.

It is probable, although not necessary, that production would expand and profits would increase until the market price began to fall; that is, until $\frac{du}{dQ}$ became negative. If this is the case, then (2:1:28) can be translated as: the maximum profit is reached at a point where the market price exceeds the marginal cost of production by an amount equal to the reduction of income occasioned by producing one more article.

We note, also, from (2:1:26) that a second differentiation results in

(2:1:30) $$\frac{d^2\pi}{dQ^2} = -\frac{d^2C}{dQ^2}\ .$$

If the profit is actually a maximum, the second derivative of π is negative or zero. When it is negative, the second derivative of C is positive. This implies that $\frac{dC}{dQ}$ is increasing at a point where the maximum profit is reached.

D. Coefficients of Production.

A coefficient of production may be defined in a number of ways but there appear to be only two ways which are economically meaningful and mathematically useful.

The first of these definitions was given by Leon Walras.* He defined the coefficient of production for a given factor to be the amount, on the average, of that factor used in the production of each finished article. Thus, $b_p = \frac{P}{Q}$, where b_p is the coefficient of production for the factor P when P units of this factor are used in the production of Q units of the finished product. This definition is valid for every point on the production surface.

The other definition was given at about the same time by Vilfredo Pareto.** He defined the coefficient of production to be the rate of change of the quantity of the factor used with respect to a change in production. Thus, $a_p = \frac{\partial P}{\partial Q}$ is the coefficient of production for the factor P.

This latter definition applies only to points along the curve of maximum efficiency and the derivative must be taken along this curve through the given point. Unless

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*See Leon Walras: "Éléments d'économie politique."

**See Vilfredo Pareto: "Manuel d'économie politique."

this is done the derivative is meaningless, since it would be without a definite direction. For example, along an isoquant it is infinite, while along a curve on which P is constant it is zero. Neither does this derivative imply that the other factors of production are held constant, since these other factors are also dependent upon Q and change when Q changes. This restriction is implied in Pareto's work although it was not clearly stated by him. However, not all authors who have followed Pareto's method have recognized this restriction.

From these two definitions it is easily seen that when the first type of coefficient is constant, the second is also constant, and that when the first varies, the second will also vary, unless P is constant.

The two coefficients may be equal at the point of minimum average cost of production, but at all other points (where they are defined) they will not, in general, be equal.

We note that the coefficient of P as defined by Walras can be computed from a knowledge of P and Q alone, using only simple arithmetic; while the coefficient as defined by Pareto requires a knowledge of all factors of production as well as a knowledge of the equation of production. With the greater ease in determination and greater generality of application, it seems more natural to use the definition of Walras.

Using either of the definitions, three classifications of these coefficients are important; namely, constant, variable and compensatory coefficients. The difference between constant and variable coefficients of production has troubled writers on the subject since its mathematical treatment was begun. A lack of clear-cut definitions and a lack of a thorough understanding of the problem have led to a number of controversies. The most famous of these is perhaps the one between Pareto and Walras. Actually, the difficulties are removed by a simple analysis of the problem.

If a coefficient of production is constant, the amount of the corresponding factor used in the production is directly proportional to the amount of the production. The amount of leather in a shoe of a certain size and type, the amount of wood in a "two by four" of a certain length, and an excise tax on production are examples of this. The case of an excise tax is perhaps the simplest one to use, although a similar argument can apply in any other case.

A tax of β on each finished commodity would not affect the technique of production and therefore would not appear in the production function

(2:1:31) $$Q = \phi(P,T,K).^*$$

However, the tax would affect the cost, hence the

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* This notation, where P, T, K represent labor, land and capital, is frequently used in analyses of this type.

cost equation becomes

(2:1:32) $$C = C_o + P\,p_p + T\,p_t + K\,p_k + \beta Q$$
$$= C' + \beta Q,$$

where p_p, p_t, p_k are the per unit prices of P, T, K, and where C' is the cost exclusive of the tax.

The average cost of production is

(2:1:33) $$u = \frac{C}{Q} = u' + \beta,$$

where u' is the average cost of production exclusive of the tax.

It is immediately evident that u attains its minimum value at the same point at which u' does, since β is constant. Hence the point of minimum average cost is unaffected by the tax.

If a profit is made, then the tax will affect the location of the point of maximum profit. The equation for the total profit is

(2:1:34) $$\pi = p\,Q - C = p\,Q - C' - \beta Q,$$

where p is the sale price. Taking Q as the independent variable, and differentiating, we have

(2:1:35) $$\frac{d\pi}{dQ} = p - \frac{dC'}{dQ} - \beta.$$

Since this derivative is zero at the point of maximum profit, we have

(2:1:36) $$\frac{dC'}{dQ} = p - \beta,$$

where $p - \beta$ is the marginal cost of production.

In general, a variable coefficient of production depends upon both the amount of production and the amounts of the various factors of production. To this general situation there are two classes of exceptions.

The first class of exceptions contains those coefficients which vary only with the total production. Factors having this type of coefficient may be defined as cost factors, since their amounts appear only in the cost equation. One example of this class is the periodic fixed cost C_o. The coefficient of this factor as defined by Walras is inversely proportional to the quantity produced. As defined by Pareto, it is zero, since the amount of the factor does not vary with a change in production. This also illustrates a case in which the two types of coefficients are never equal, even at the point of minimum average cost.

The second class of exceptions contains those coefficients of production which are not directly dependent upon the quantity of production. The only components of this class which are of interest here are the coefficients of Walras, as determined on a linear homogeneous surface. Such coefficients depend entirely upon the proportions of the various factors of production and not upon their specific amounts. In this way, these coefficients are constant for constant proportions, and hence are independent of the actual amount of production.

If, when the quantity of one technical factor of production is reduced, the quantity of another can be increased in such a manner that the production remains the same, the two factors are regarded as substitutable. In terms of their coefficients, the reduction of one coefficient can be "compensated" by the increase of another.

Many authors have attempted to formulate this situation in mathematical terms. None thus far has succeeded for the coefficients as defined by Walras. Pareto assumed his compensatory relation as one of his basic equations. The difficulty with the coefficients of Walras is that no such single equation exists, although it is easy to explain in mathematical terms what really is meant by "compensation."

Walras assumed outright that

(2:1:37) $$\phi(b_p, b_t, b_k) = 0$$

was such a relation. Other authors have pointed out correctly that this must be a different function from the production equation $Q = \phi(P,T,K)$.

Actually, the above equation should have been written as

(2:1:38) $$\phi(Qb_p, Qb_t, Qb_k) = Q .$$

This is merely the production equation with P, T, K replaced by their values occuring in the equations defining

the coefficients of production. It explains nothing and leads nowhere unless additional assumptions are made.

Walras, in effect, assumed his coefficients to be independent of Q. This assumption has two immediate consequences. First, for Q = 1, his defining equation should have read $\phi(b_p, b_t, b_k) = 1$. Thus, when one replaces zero on the right, the function is the same as in the production equation. Second, the assumption of independence requires that the production function be linearly homogeneous.

A few of the later authors have assumed that the coefficients satisfy a relation independent of the production function. They evidently have not examined the consequences of such an assumption. In the first place, a third relation (the cost and production equations are the other two) between Q, P, T, K introduces one more dependent variable and invalidates all of the preceding mathematical development. In the second place, there is no economic reason for assuming that the compensatory relations cannot be derived from the production function.

This compensatory feature among the coefficients of production as defined by Walras refers to the property of the production function along a contour line or isoquant. Along this curve, the following equation holds:

(2:1:39) $\phi(P,T,K) = \phi(P+\Delta P, T+\Delta T, K+\Delta K) = Q.$

This relation involves the delta-increments, or the actual changes, of all quantities which are allowed to vary. For all production surfaces considered here, at least two of the increments are different from zero and are opposite in sign.

The coefficients of P and T at the first point are $b_p = \frac{P}{Q}$ and $b_t = \frac{T}{Q}$. At the second point they are $b'_p = \frac{P+\Delta P}{Q}$ and $b'_t = \frac{T+\Delta T}{Q}$. If we assume that ΔP and ΔT are the only two increments different from zero, then we know that they are opposite in sign and that the decrease in one coefficient has been compensated by an increase in the other. In this manner, the increase in one set of coefficients is compensated by a decrease in one or more of the others. Still other coefficients may remain unchanged. This type of compensation is always possible unless the contour line at this level of production reduces to a point.

The compensatory feature among the coefficients as defined by Pareto is embodied in his basic equation in which he assumes that

(2:1:40) $$f(a_p, a_t, a_k) = 0.$$

Nowhere does he attempt to justify the existence of such a function or to explain its economic significance. Indeed, any such explanation seems to be related to the production function of Walras.

If we begin with the production function $Q = \phi(P,T,K)$ and restrict the changes in P, T, K to the curve of maximum efficiency, as we must in the case of Pareto's coefficients, we have

(2:1:41) $$1 = \frac{\partial\phi}{\partial P}\frac{\partial P}{\partial Q} + \frac{\partial\phi}{\partial T}\frac{\partial T}{\partial Q} + \frac{\partial\phi}{\partial K}\frac{\partial K}{\partial Q}.$$

The above equation can be written in the form

(2:1:42) $$\phi_P a_p + \phi_T a_t + \phi_K a_k - 1 = 0.$$

Now, P, T, K are functions of Q alone, hence we reduce this last equation, by substituting for P, T, K in the partial derivatives of ϕ, to the form

(2:1:43) $$\psi(a_p, a_t, a_k, Q) = 0.$$

This must be equivalent to, if not identical with, the equation assumed by Pareto. If it were not, there would be a third equation, as in the case of Walras, which would invalidate all of the mathematical work on production by Pareto. Furthermore, it is apparent that the presence of Q must be provided for in the defining equation. Only in the case of a linear homogeneous production function will it be absent.

E. Limitational Conditions.

Up to this point, the technical factors of production have been regarded as independent of one another and usable in any proportions. However, in actual practice, such is not the general case. In this section we will

investigate certain limitational conditions and demonstrate how they may be handled mathematically.

Limitational conditions may be divided into two large classes. The conditions of the first class place upper or lower limits upon the amounts of one factor with respect to another factor or with respect to the total production. These relations are expressible in mathematical terms as inequalities. For example, the number of units of the X-factor might never be less than the number of units of the Y-factor; then $x \geqq y$ would express the relationship between them.

For a slightly different illustration, assume that a certain iron ore contains forty percent pure iron. Then the number of pounds of pure iron produced could never exceed four-tenths the number of pounds of ore used. This is expressible as $z \leqq 0.4\ x$. Also, it might not be profitable to extract all of the iron from the ore; in this case, z is actually less than $0.4\ x$.

The conditions of the second class are expressible in an exact relation between two or more of the quantities involved. This relationship is mathematically stated with an equation. As in the first class, the relationship may be between a factor of production and the amount produced or it may be between two or more factors.

The case in which a factor is related to the total production has been discussed under coefficients of

production (page 92 ff.). Its effect upon the optimum point of production enters the problem by way of the cost equation. When the relationship is known, the problem is generally determinate.

Another type of limitational case belonging to this class is that in which one factor is related to another by a definite equation. Assume the production function to be $z = f(x,y,w)$, where w is a factor whose amount is related to that of x by the equation $w = h(x)$. The cost equation is

(2:1:44) $$C = x\,p_x + y\,p_y + w\,p_w.$$

By means of the relationship $w = h(x)$, w can be eliminated from both the production equation and the cost equation. They become

(2:1:45) $$z = \phi(x,y) \quad \text{and}$$

(2:1:46) $$C = x\,p_x\left[1 + \frac{h(x)}{x}\,\frac{p_w}{p_x}\right] + y\,p_y.$$

The sole effect of the W-factor is to change the price of the X-factor. If the quantity in the brackets of (2:1:46) is constant, the new price is constant; if it is variable, the new price is variable in the manner indicated.

The existence of such a relationship between two factors merely complicates the problem without altering the methods of solution. If w had been a function of both x and y, its elimination would have been possible

in a similar fashion, with the price of x dependent upon the amount of y.

If a factor, such as w, is related by an equation to both the total production, z, and the amounts of one or more other factors, the problem is still determinate. Together with the production function, this new relation gives us two equations which must be satisfied. The solution of the problem of minimum average cost or maximum profits must lie on the intersection of the two surfaces represented by the two equations.* Although this third equation adds to the number of given equations, it reduces by one the number of derivatives which are zero at the maximum or minimum. Thus the problem remains determinate as before.

The existence of inequalities between various factors of production serves to divide the space into compartments. In some of these $z = 0$, or there is no production. Suppose the problem is one of determining the minimum average cost of production. Each of the compartments in which production is different from zero must be examined in the usual way for a minimum. In addition, the curves on the production surface cut by the partitions between compartments must be examined; and finally, an examination must be made of the points where two such curves intersect.

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*In general, this statement refers to a space of n dimensions, in which n is equal to the number of independent variables.

It is possible that somewhere along the boundaries the average may be less than at any other point, yet the derivatives may not be zero there. Among the values found, we select the smallest one. If maximum profit is desired, the procedure is the same, except we now select the largest value.

If the limitational inequalities involve the total production, these conditions serve to break the production surface into facets. Each facet has a separate equation for the production in the corresponding compartment. Here an examination will have to be made of each facet, each edge between two facets, and each corner where two or more edges come together. The maximum or minimum, as desired, is then selected from among all solutions.

For illustrative purposes we will impose a limitational condition upon the example of section B (p. 78 ff.). We have the production surface defined by

$$z = \sqrt{100 - (x-10)^2 - (y-10)^2}$$

when x and y are on or within the circle

$$(x-10)^2 - (y-10)^2 = 100$$

in the xy-plane, and $z = 0$ for all other values of x and y.

We note that the very statement of the problem imposes limitational conditions since the production

surface has two separate equations. In the strictest sense, the existence of any production surface at all is a limitational condition which prevents unlimited production from a given amount of x and y. One of the assumptions made was that every producer adjusted his methods so as to realize the limit. This, of course, does not always exist in reality.

We now impose the limitational condition that $z \leqq 2x$. In effect, this limitation states that z cannot exceed $2x$ although it may be equal to that amount. This condition adds a third facet to the production surface.

Beginning with a solution similar to that of page 79, we have:

for $p_x:p_y = 1:1$, $\quad x_o = y_o = 5, \quad z_o = 5\sqrt{2}$;

for $p_x:p_y = 3:7$ $\quad x_o = 7, \quad y_o = 3, \quad z_o = \sqrt{42}$;

and in general,

$$p_x:p_y = y_o:x_o, \qquad x_o + y_o = 10, \quad z_o = \sqrt{2x_o y_o} \ .$$

We may interpret the limitational condition, $z \leqq 2x$, as follows:

The plane $z = 2x$ intersects the production sphere in a circle whose projection downward upon the xy-plane is the ellipse

$$\frac{(x-2)^2}{4} + \frac{(y-10)^2}{20} = 1 \ .$$

This ellipse is cut by the line $x+y=10$ at the points $(\frac{10}{3},\frac{20}{3})$ $(0,10)$ and $(0,10)$.

If the ratio of the p's is ≤ 2, the point of tangency will be on the sphere. If the ratio is > 2, the point of maximum production will be along the intersection of $z = 2x$ and the production sphere.

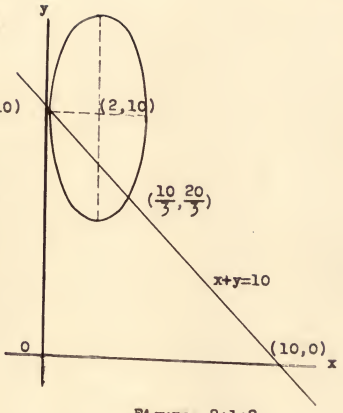

Figure 2:1:2

It can be shown that, in this case, the point will generally remain at $(\frac{10}{3},\frac{20}{3})$.

The critical ratio of the prices in order to make the restriction effective is $p_x:p_y = 2:1$.

To illustrate, we take the ratio as $p_x:p_y = 4:1$. We now have

$$(x-10)^2 + (y-10)^2 + z^2 = 100$$

$$z = 2x$$

$$4x + y - \mu z = 0,$$

where μ must be chosen so there is only one intersection.

We have

$$y = 2\mu x - 4x = 2x(\mu - 2),$$

$$(x-10)^2 + (2\mu x - 4x - 10)^2 + 4x^2 = 100,$$

$$x^2(21 - 16\mu + 4\mu^2) + x(60 - 40\mu) + 100 = 0.$$

In order for x to have only one value

$$(60 - 40\mu)^2 - 400(21 - 16\mu + 4\mu^2) = 0, \quad \text{or}$$

$$1600\mu = 4800, \quad \text{and}$$

$$\mu = 3.$$

Thus,

$$x^2(21 - 48 + 36) + x(60 - 120) + 100 = 0, \quad \text{or}$$

$$9x^2 - 60x + 100 = 0, \quad \text{and}$$

$$x = \frac{10}{3},$$

$$y = \frac{60}{3} - \frac{40}{3} = \frac{20}{3}.$$

CHAPTER 2

CONSUMPTION IN A STATIC ECONOMY

Most of the early investigators of the problem of consumption, or consumer demand, found it necessary to assume the existence of a "utility function" for the individual; i.e., that the individual derives a certain amount of utility from each of the commodities that he acquires, and that the total utility can be expressed as a function of the commodities acquired. They further assumed that the individual spends his income in such a manner as to secure the maximum total utility. Briefly, their arguments were predicated upon the principle of diminishing marginal utility and the conclusion that the marginal utilities of the commodities purchased must be proportional to their respective prices.

This approach has much to recommend it and many modern authors use the same type of analysis. However, the concept is psychic in character and objective measurement is impossible. In an effort to evolve a more satisfactory approach to the problem of consumption Edgeworth, Pareto, Walras and others developed the concept of indifference curves or indifference surfaces. Thus, instead of investigating the relation between the curve of diminishing marginal utility and the demand curve, these economists turned their attention to the problem

of related (complementary and competitive) commodities.

In this chapter we will discuss both types of analyses in some detail and will consider certain important variations of the basic assumptions.

A. Maximization of Utility.

The problem in which we are interested here is, in its simplest form: if an individual with a given income is to spend it for goods in a market in which the prices of the goods are fixed, how shall he divide his expenditures?

Several basic assumptions are immediately apparent. First, the time interval is of secondary importance here and may be taken as given for the problem. Second, the individual has a fixed income for this time interval and spends his entire income for the commodities of his choice. Third, the quantities of the commodities are unlimited as far as the individual is concerned. Fourth, the prices of the commodities are given and remain fixed for the time interval. Fifth, the individual's utility function is known and does not change during the time interval. Sixth, the individual maximizes his utility subject to his budget equation restriction. Seventh, certain conditions of divisibility and substitutability must exist in order for mathematical uniqueness, continuity, etc., to exist.

The fifth and sixth assumptions merit further comment. The concept of utility or preference is an ordered but non-measurable one. If a consumer purchases only two commodities, X and Y, then writing $u = f(x,y)$ is only one way of indicating the dependence of his utility upon the amounts x and y of the commodities that he acquires. In general, if $F(u)$ is any monotonic increasing function of u, then $F(u) = F\left[f(x,y)\right]$ will serve as an index of the consumer's utility. Thus, $u = f(x,y)$, $u = \left[f(x,y)\right]^3$, $u = \log f(x,y)$, etc., will serve equally well.*

In this section we will also employ marginal utility. If $u = f(x_1,\ldots,x_n)$ is one form of an individual's utility function for the commodities $X_1,\ldots,X_n$, it might appear that $\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n}$ represent the marginal utilities of the various purchases; e.g., that $\frac{\partial u}{\partial x_1}$ measures the rate of change of utility when the purchase of the X_1-commodity changes, the amounts purchased of the other $n - 1$ commodities remaining unchanged. However, since utility is a non-measurable quantity and can be indicated by $F(u) = F\left[f(x_1,\ldots,x_n)\right]$, it is dangerous to attach meanings to increments of utility.

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*See R. G. D. Allen: "Mathematical Analysis for Economists," pp. 276-278.

We have that $\frac{\partial F}{\partial x_1} = F'(u)\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} = F'(u)\frac{\partial u}{\partial x_n}$, and marginal utilities involve the derivatives of the arbitrary function F; hence they lose their definiteness. However,

$$\frac{\partial F}{\partial x_1} : \frac{\partial F}{\partial x_2} : \ldots : \frac{\partial F}{\partial x_n} = \frac{\partial u}{\partial x_1} : \frac{\partial u}{\partial x_2} : \ldots : \frac{\partial u}{\partial x_n},$$

and the ratios of the marginal utilities are definite concepts independent of the non-measurability of utility.

We now consider the case of an individual who has a known utility function

(2:2:1) $$u = u(x_1, \ldots, x_n),$$

where $x_1, \ldots, x_n$ are the amounts of the various commodities which the individual buys and whose prices per unit are fixed at $p_1, \ldots, p_n$.

We assume that the individual has a given income M and that he spends his entire income on the commodities. We have for his budget equation

(2:2:2) $$M = p_1x_1 + \ldots + p_nx_n.$$

We now assume that the individual maximizes his utility subject to his budget restriction. We form the function

(2:2:3) $$w = u + \mu\left[M - \sum_{i=1}^{n} p_i x_i\right],$$

where μ is a Lagrange multiplier. Since u and w will attain their maximum values together, we have, from the necessary conditions for a maximum, that

(2:2:4) $$\frac{\partial u}{\partial x_i} - \mu p_i = 0, \qquad i = 1,\ldots,n.$$

Letting u_i represent $\frac{\partial u}{\partial x_i}$, we write (2:2:4) in the form

(2:2:5) $$\frac{u_1}{p_1} = \frac{u_2}{p_2} = \cdots = \frac{u_n}{p_n} = \mu .^*$$

If $\frac{\partial u}{\partial x_j}$ is less than μp_j, then the individual will purchase none of the j th commodity and x_j will not enter his utility function.

The translation of equation (2:2:5) into economic language leads to

Theorem 2:2:1. If an individual maximizes his utility subject to his budget restriction, the marginal utilities of the commodities entering his utility function are proportional to the respective prices.

We have n equations from (2:2:5) and one equation from (2:2:2) which, in general, will enable us to solve for the $n + 1$ variables $x_1,\ldots,x_n,\mu$.

We arrive at similar conclusions by regarding the problem as one in which the individual minimizes his expenditures for a given level of utility.

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*A number of authors have regarded μ as constant for all prices. There is no basis for this. Some define μ as the marginal utility of money. This is true only if money is one of the x_i which enters the utility function. Such a contingency leads to extreme complexity. See P. A. Samuelson: "Studies in Mathematical Economics and Econometrics."

To determine the second order, or sufficient, conditions (where $d^2u < 0$) we find that the determinants

$$(2:2:6) \qquad \begin{vmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{12} & u_{22} \end{vmatrix}, \ldots, \begin{vmatrix} 0 & u_1 & \cdots & u_n \\ u_1 & u_{11} & \cdots & u_{1n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ u_n & u_{1n} & \cdots & u_{nn} \end{vmatrix}$$

are alternately positive and negative.*

B. Indifference Curves and Surfaces.

The basic assumption in the indifference analysis is that utility is constant for the period of time under consideration. There are several implications of this assumption which have not always been specified and seldom made clear in the literature on the subject. The first portion of this section will deal with the behavior of indifference curves and surfaces and the actual assumptions necessary to produce this behavior.

An indifference curve is usually defined as follows (See figure 2:2:1): if an individual is indifferent to the combination of goods represented by the coordinates P and Q; i.e., if, when offered a free choice, he would be equally satisfied with one combination or with the other, then P and Q lie on the same indifference curve; and

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* See Lemma 1:1:4 and J. R. Hicks: "Value and Capital," p. 306.

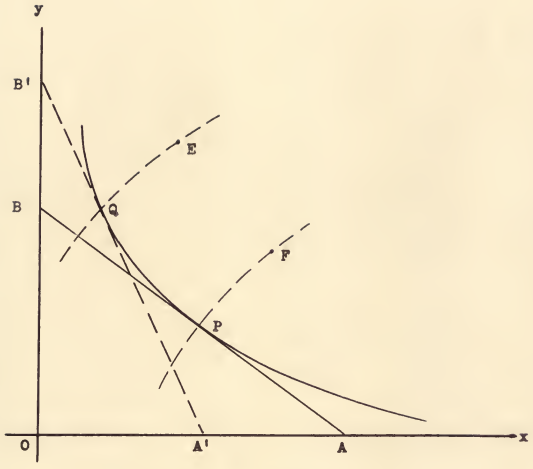

Figure 2:2:1

the totality of such points, or the locus of all such points, defines the indifference curve. In the case of n goods, the locus of such points will, in general, define a hypersurface.

The line AB in the figure is the individual's budget line. It is tangent to his indifference curve at the point P. The equation of the line AB is

(2:2:7) $$x p_x + y p_y = M ,$$

where x and y are the amounts of the two commodities in question, p_x and p_y their respective prices, and M the individual's known income. The parameters here are p_x, p_y and M.

By the definition of the line AB, the cost of any combination of the two commodities (represented by a point on the line) is the same from point A to point B, and this cost is M. The prices are such that the income (or expenditure) will purchase the amount OA of X or the amount OB of Y if the entire income is spent for one commodity.

Thus, an individual with an income M can purchase any combination of goods within the triangle AOB. If we assume that more of a good is always desirable, then the equilibrium point of the individual will fall somewhere on the line AB, rather than at an interior point of the triangle. Now, even though P is the equilibrium point for the individual, any combination of goods along the line AB would be equally acceptable to him provided he has the privilege and opportunity of exchanging what he receives for what he wants. Hence, in order for the equilibrium point to have uniqueness, we must impose the following condition:

Condition 2:2:1. The quantities of goods bought by the individual must be consumed by him; i.e., he cannot sell or exchange them.

The assumption of uniqueness imposes a restriction upon the indifference curve PQ of Figure 2.2.1. All points except P must lie outside the triangle AOB. This holds for every point which can be an equilibrium point and leads to a characteristic which we state as

Theorem 2:2:2. An indifference curve or surface must be convex with respect to the origin.

We now determine the point P subject to two conditions; namely, that the prices have a given ratio (which fixes the slope of the budget line) and that the income be sufficient to purchase the combination of goods represented by the point P. Therefore, the combination of goods represented by the point Q of Figure 2:2:1 would become the equilibrium point, albeit equally desirable as point P, if and only if (i) the prices are such as to make the budget line have the slope of the line tangent to the indifference curve at Q, and (ii) the income is such that the combination of goods at Q can be purchased.

If the prices of the commodities remain fixed and the income changes, then the individual will move along the "expansion paths" indicated by the lines PF or QE.

The individual will choose the point P if two things are exactly right; i.e., the ratio of p_x to p_y,

which determines PF, and an income large enough to reach the point P on the expansion path PF. Therefore, the point Q would become a new equilibrium point only if the following condition is satisfied:

Condition 2:2:2. A point Q on the indifference curve PQ is equally as desirable as the point P on the curve only if (i) the relative prices are changed to the ratio which determines the expansion path QE, and (ii) the individual's income is sufficient to reach the point Q on QE.

It is possible that the income sufficient to reach a point F on the expansion path PF might be attained by the expenditure of greater effort on the part of the individual and that the additional goods represented by the point F might exactly compensate for the greater effort. In this case, the combination of goods at F would be equally desirable as the combination at P. In order to eliminate this contingency we must impose the following restriction:

Condition 2:2:3. Only those combinations of goods which can be obtained for the same effort on the part of the individual are considered equally desirable.

There are two other important characteristics of indifference curves. We state these as the following

theorems:

Theorem 2:2:3. Indifference curves do not intersect.

Theorem 2:2:4. The slope of an indifference curve must be negative.

The preceding conditions and theorems are perfectly general and can easily be extended to indifference surfaces of any number of dimensions.

In the foregoing discussion we have examined some of the geometric properties of indifference curves and surfaces, observing what properties were necessary if certain results were to be attained. We will now investigate the possibility of deriving the equations of such curves and surfaces.

Theoretically, there are many experiments which could be performed to determine indifference curves or surfaces. Points on the same surface could be found in the following manner: (i) Consider certain specific amounts $^1x_1, ^1x_2, \ldots, ^1x_n$, of the various commodities; these will determine a certain level of utility. (ii) Determine certain other combinations, $^2x_1, ^2x_2, \ldots, ^2x_n$, to which the individual under consideration is indifferent. Denote the first point by P and the second point by Q. When the entire surface is known the slope in every direction at the point P is known. If the amount of

the commodity $^1x_1$ is greater than zero we can find all the prices of the other commodities relative to p_1 by the slopes with respect to two axes at a time. Knowing the relative prices and the amounts of the commodities, we can find the income necessary to purchase these commodities, but only relative to the prices.

The above method establishes the indifference surfaces directly and without any intermediate calculation. The following method will establish indifference curves or surfaces indirectly. It will also emphasize the fact that utility is non-measurable.

We let the individual be given a certain income which he is to spend for consumption goods at fixed market prices. He will reach an equilibrium point at which we assume he spends his entire income.

We now let the income be changed, the prices remaining fixed. The locus of successive equilibrium points will lie on a one-dimensional curve in n-dimensional space. At each point on this curve the budget plane has a definite orientation; this orientation will be the same at all points since it is determined by the relative prices.

For a different set of prices there will, in general, be another expansion path. At all points of this latter path the budget plane will have the same orientation in

space, an orientation, however, which is different from that at points along the first expansion path. In this way we determine a path for each set of relative prices.

Before indicating the equations involved in the general solution we will develop the simple case of two commodities.

We assume that an individual purchases amounts x and y of commodities X and Y and that the amounts purchased are inversely proportional to their prices, p_x and p_y. We have

(2:2:8) $$\frac{y}{x} = \frac{p_x}{p_y} = a, \quad \text{or} \quad y = ax.$$

The above equation expresses the ratio of the purchases for any income. We define the income I by the following equation:

(2:2:9) $$I = x\,p_x + y\,p_y.$$

Since (2:2:9) is the equation of a straight line, it can contain only two independent parameters. By dividing this equation through by p_y we have

(2:2:10) $$ax + y = c.$$

In the above equation the parameter c is the number of units of the Y-commodity which could be purchased if the individual spent all of I on that commodity.

Equations (2:2:8) and (2:2:9) must hold simultaneously

and their solution yields

(2:2:11) $$x = \frac{c}{2a}, \quad y = \frac{c}{2}.$$

Thus, whenever a and c are given, x and y can be determined. This means that whenever the prices of the two commodities are given and the income relative to these prices is known, the amounts of each commodity purchased can be determined.

Equation (2:2:11) states that the individual will spend exactly half of his income on each commodity. It should be noted that the second equation of (2:2:11) is independent of the prices or their ratio. It represents a curve along which c is constant and a is variable. We designate these curves as a-paths or income-paths.

Similarly, equation (2:2:8), where $y = ax$, represents a curve along which a is constant and c is variable. We designate these curves as c-paths or expansion-paths.

In effect, the two families of curves constitute a second coordinate system superimposed upon the xy-plane. By constructing lines as shown in Figure 2:2:2, page 120, we are able to read from one set of coordinates to the other.

Two other sets of curves may be constructed in the xy-plane; i.e., budget lines and contour lines. One budget line (the line AB, drawn for $a = 2$ and $c = 10$)

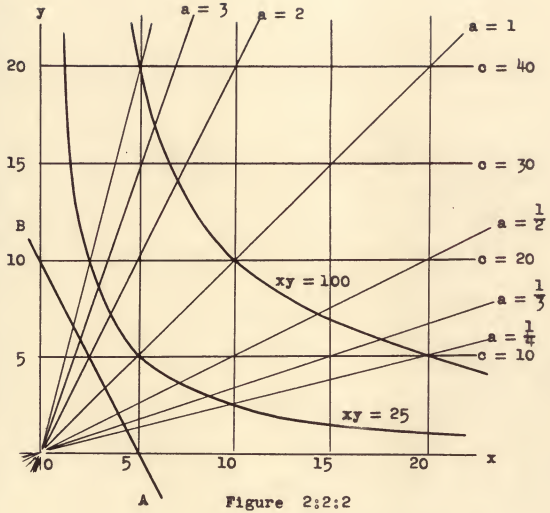

Figure 2:2:2

a-paths: $y = \frac{c}{2}$

c-paths: $y = ax$

budget lines: $ax + y = c$

contour lines: $xy = k$

is shown in Figure 2:2:2. Its equation is $2x + y = 10$. This line crosses the lines $a = 2$ and $c = 10$ at their intersection. Because of this fact, the graph of the budget line is quite unnecessary since it supplies no additional information. However, the equations of the budget lines are needed. By using these equations, together with the equations of either the c-paths or the a-paths, we are able to determine the equations of the contour lines.

Contour lines are defined as being tangent to the budget lines at points corresponding to the given values of a and c.

We take the equation of the family of contour lines as

(2:2:12) $$\phi(x,y) = k.$$

Since the above curves are tangent to the budget lines, we have for their slope

(2:2:13) $$\frac{dy}{dx} = -\frac{\phi_x}{\phi_y} = -a.$$

If the parameter a is eliminated between equations (2:2:8) and (2:2:13) the result is the differential equation

(2:2:14) $$\frac{dy}{dx} = -\frac{y}{x}.$$

Solving this equation, we have

(2:2:15) $$\phi(x,y) = xy = k.$$

Here k is the arbitrary constant of integration whose value is determined by the coordinates of the point through which the curve is to pass.

The two contour lines shown in Figure 2:2:2 are $xy = 25$ and $xy = 100$. The contour line tangent to the budget line shown would have to pass through the point $(2.5,5)$ and its equation would be $xy = 12.5$.

In attempting to formulate the equations of contour lines or indifference curves we would need the following information: (i) for each possible price ratio a, the statistical values of x and y for each income ratio c; or (ii) for each possible income ratio c, the statistical values of x and y for various price ratios a. Having this information, we could construct the parametric equation of the corresponding family of curves. The budget equation will always be available and the equations of the contour lines can be determined.

The individual maximizes his utility by deciding upon how he will spend each given income. This maximum is an observed statistic. Thereafter, no maximum or minimum is involved. Also, the solution is direct unless at some point the mathematics becomes such that the problem cannot be developed step by step. This is equivalent to stating that the individual cannot make up his mind or changes it so frequently as to make the solution impossible.

It might be well here to make an observation regarding one of the basic assumptions of the problem. Most authors refer to this as the "assumption of rational behavior." Actually, the rationality, morality or intelligence of the behavior has no bearing on the mathematics of the problem. What is really meant is that we must assume consistent behavior on the part of the individual; i.e., we must assume that for the period of time under consideration the individual when confronted with a given set of circumstances will always react in the same manner.

We will now extend the methods of the two-dimensional case to the general case. The budget equation will contain $n + 1$ parameters; i.e., the n prices and the income. This number can be reduced to n by dividing the budget equation through by one of the prices. We must be sure that the price used is different from zero. We could, of course, divide through by the income, since if that were zero the problem would be trivial. However, we shall find it more convenient to divide by one of the prices. Since all of the prices cannot be zero we may, without loss of generality, label the commodities so that $p_1 \neq 0$. The advantage gained by regarding one of the commodities as a numeraire is that the prices of the other $n - 1$ commodities can be expressed relative to this one, and

these relative prices will retain the same value all along the same expansion path.

We let $a_i = \frac{p_i}{p_1}$ denote the prices relative to the price of the commodity X_1. The equations defining the expansion path can be written in the form

(2:2:16) $$f^{(i)}(x_1,\ldots,x_n,a_2,\ldots,a_n) = 0, \quad i = 2,\ldots,n.$$

The above equations define $n - 1$ surfaces in n-dimensional space and thus determine a one-dimensional curve in this space. This curve is the expansion path. If certain conditions are satisfied (and we here assume them to be satisfied) we may solve for the $n - 1$ a's in terms of the n x's. From (2:2:16) we have

(2:2:17) $$a_i = g^{(i)}(x_1,\ldots,x_n), \qquad i = 2,\ldots,n.$$

We now assume the existence of the indifference surfaces and further assume that they are of the form

(2:2:18) $$\phi(x_1,\ldots,x_n) = k,$$

where k is an arbitrary constant.

From previous development we have that

(2:2:19) $$\frac{\phi_i}{\phi_j} = \frac{p_i}{p_j} = \frac{a_i}{a_j}$$

at each equilibrium point. These equilibrium points are exactly those on the expansion path. Thus, equations (2:2:19) may be used to eliminate the a_i. We then have

(2:2:20) $$\frac{\phi_1}{1} = \ldots = \frac{\phi_i}{g^{(i)}} = \ldots = \frac{\phi_n}{g^{(n)}} .$$

From equation (2:2:18) we obtain the total differential of ϕ as

(2:2:21) $$d\phi = \phi_1 dx_1 + \ldots + \phi_i dx_i + \ldots + \phi_n dx_n = 0.$$

If the individual has a family of indifference surfaces, then the differential equation

(2:2:22) $$dx_1 + g^{(2)}dx_2 + \ldots + g^{(i)}dx_i + \ldots + g^{(n)}dx_n = 0,$$

formed by combining the equations of (2:2:20) and (2:2:21), has an integrating factor. When this factor is found the solution of (2:2:22) will yield the general equation of the indifference surfaces.

This solution will contain an arbitrary constant. Also, once an integrating factor has been found for the differential equation, an indefinite number of them can be found.

The value of the constant is the value of the particular utility index. Since there are no boundary conditions to determine this constant for each point, or even anything to indicate which of the various solutions to use, the measurability of utility is indeterminate. One point which should be noted here is that mathematically the constant of integration is not required to increase

with income although economically it is usually assumed to do so.

C. Variations of the Basic Problem.

The effect on demand of certain changes in the parameters has been considered by several investigators.* For the most part, their analyses have been based on an equation known as "the fundamental equation of value." In one form it is written

$$(2:2:23) \qquad \frac{\partial x_s}{\partial p_r} = - x_r \frac{\partial x_s}{\partial M} + x_{rs}, \qquad s = 1,\ldots,r,\ldots,n.$$

The symbols of the above equation refer to n goods $(X_1,\ldots,X_n)$ bought at the prices $p_1,\ldots,p_n$, in the amounts $x_1,\ldots,x_n$ by an individual with an income M and a utility index $u(x_1,\ldots,x_n)$, the value of which depends upon the amounts purchased. In particular, the x_s are the coordinates of the equilibrium point where u is a maximum for a given M. The equation itself expresses the rate of change in the amount of one of these goods with respect to a change in its own price $(s = r)$ or the price of another $(s \neq r)$ under a given set of conditions. In

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*See J. B. Hicks: "Value and Capital," pp. 307 ff.

Henry Schultz: "The Theory and Measurement of Demand," pp. 37 ff.

J. L. Mosak: "General Equilibrium Theory in International Trade," pp. 19 ff.

_____.: "Studies in Mathematical Economics and Econometrics," pp. 69 ff.

this form but with slightly different notation, the equation was derived by Slutsky.* His work did not gain general recognition until it was publicized by later writers.

The treatment of this equation and of others related to it has been inadequate from a mathematical point of view. Also, the geometric interpretations of these equations appearing in the economic literature have been unsatisfactory.

In the first place, there is some question as to whether or not this equation is more "fundamental" than certain other ones which we shall derive and which are related to it. In the second place, the meanings of the equations are not clear largely because of the use of inadequate notation in their derivation and because the conditions under which they are derived and manipulated are not stated exactly. In the third place, all of the diagrams which have come to the attention of this author pertain to a related but entirely different mathematical problem.

In this section we will clarify the derivations of the equations involved and interpret the meanings of the terms of the equations.

There are two basic situations for which an equilibrium

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* Eugen Slutsky: "Sulla teoria del bilancio del consumatore," Giornale degli economistri, LI (1915), pp. 1-26; translated by Pietro Castiglioni, University of Florida, (1949).

point can be determined. The first of these has been emphasized almost to the exclusion of the second although both are equally necessary for a clear understanding of the details involved.

The first basic situation relates to an individual who has a fixed income M and a given index of utility u, and who enters the market where the n goods $X_1,\ldots X_1$ are on sale at the prices $p_1,\ldots,p_n$. Since the entire income is assumed to be spent on goods, the quantity M might equally well be designated as the total expenditure of the individual. The problem is to determine the amounts $x_1,\ldots x_n$ which the individual will purchase in order to maximize his utility.

As we have shown in section A of this chapter, these amounts are obtained from the solution of the $n + 1$ equations

$$(2:2:24)\qquad \begin{aligned} &\sum_{i=1}^{n} p_i x_i = M \\ &u_s - \lambda p_s = 0 \ , \end{aligned} \qquad s = 1,\ldots,n.$$

The solutions of these equations may be written symbolically as

$$(2:2:25)\qquad x_s = \alpha_s(p_1,\ldots,p_r,\ldots,p_n,M), \quad s = 1,\ldots,n.$$

These are the coordinates of the point of maximum utility expressed as functions of the parameters of the problem. This point, the equilibrium point, is the point P of

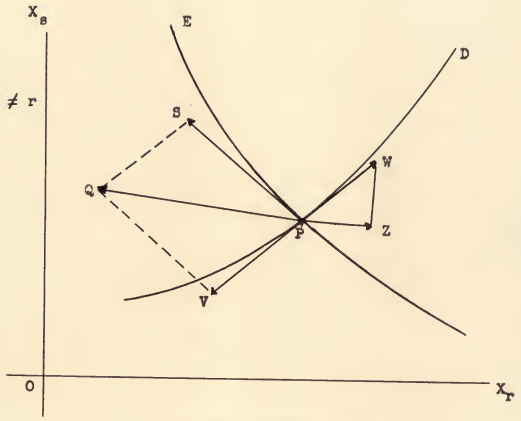

Figure 2:2:3

Figure 2:2:3.

The second basic situation relates to an individual entering the same market as before but who wishes to maintain a constant level of utility while keeping his total expenditure M at a minimum. The coordinates of the equilibrium point under these conditions are given by the solution of the set of n + 1 equations

$$u(x_1,\ldots,x_n) = C$$

(2:2:26)

$$u_s - \lambda p_s = 0, \qquad s = 1,\ldots,n.$$

The solutions of these equations for the x_s may be written symbolically as

$$(2;2;27) \qquad x_s = \beta_s(p_1,\ldots,p_r,\ldots,p_n,C), \qquad s = 1,\ldots,n.$$

These solutions are different from those in (2;2;25) and have been so designated. They contain the parameter C instead of M and are different functions of the prices, as we shall see. There are, however, several relationships between the two sets, especially between their derivatives. This will be shown in the special variations which follow.

The mathematical problem with which this section is concerned is the determination of the direction and rate of movement, or the direction and amount of movement, in the equilibrium point as one of the parameters changes while the other parameters remain constant or have certain conditions imposed upon them.

We note that the parameters of the problem are the prices and either the income M or the index of utility C, but never both M and C simultaneously. In the variations which follow the dependent variables will in every case include the x_s, all of which change under various sets of assumptions. Since we will have only one independent variable in the following discussions, the derivatives of the x_s and of any dependent parameters will be written as total derivatives. The derivatives of the α and β functions, however, must be indicated by

partial derivatives of these functions. This accounts for the dual notation used for the derivatives.

There are three variations of the basic problem which are so simple that they too may be considered as basic. They are more elementary than the variations of the "fundamental equation" itself, since the latter variations are nothing more than combinations of the simple ones. It is this fact that gives rise to the question of whether or not the designation "fundamental" is appropriate for the equation to which it is now attached.

Variation (i). The first of these simple variations is the variation of the parameter M in the equations of (2:2:25). As M varies, the locus of the equilibrium point P is a one-dimensional curve in n-space. We designate this as an expansion path. Along such a curve the prices remain fixed while M and C continue to vary. One such expansion path is the curve PD of Figure 2:2:3.

The rate at which P moves along the locus with respect to the change in M is determined by finding the rate at which the coordinates of P vary with M; that is, by finding $\frac{dx_s}{dM}$ for each value of s. We have

$$\frac{dx_s}{dM} = \frac{\partial \alpha_s}{\partial M}, \qquad s = 1,\ldots,n. \tag{2:2:28}$$

The forms of the α functions are unknown. Therefore the forms and characteristics of the partial deriva-

tives in (2:2:28) would remain unknown also if it were not possible to find these partial derivatives by another method. We make use of a theorem from the theory of functions which states, "With proper conditions of continuity, the derivative of the solution of a set of simultaneous equations is equal to the solution of the set of differentiated equations."

This theorem enables us to take the derivatives in (2:2:24) and then solve for each derivative with respect to M. By this means, the values are found to be

$$\frac{dx_s}{dM} = \frac{\partial \alpha_s}{\partial M} = \frac{\lambda U_s}{U}, \qquad s = 1,\ldots,n. \tag{2:2:29}$$

where U is the determinant

$$\begin{vmatrix} 0 & u_1 & u_2 & \cdots & u_n \\ u_1 & u_{11} & u_{12} & \cdots & u_{1n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ u_n & u_{1n} & u_{2n} & \cdots & u_{nn} \end{vmatrix},$$

and where U_s is the cofactor of u_s in U.

Since the derivative here denotes a rate of movement, it can be represented by a vector. As a vector, each derivative in (2:2:29) lies along, or parallel to, the corresponding axis. Each derivative represents the component of the rate in the movement of P along this axis. Therefore, the total rate of movement is the vector sum

$$\sum_{i=1}^{n} \frac{dx_i}{dM} = \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial M} = PW, \tag{2:2:20}$$

and where PW is the vector shown in Figure 2:2:3.

This vector PW has the direction of the tangent to the curve PD at the point P. Its sense is determined by the sense in which P moves as M increases; and its magnitude is the square root of the sum of the squares of the magnitudes of its components.

In the two-dimensional diagram of Figure 2:2:3, PZ is parallel to the X_r-axis and is equal in magnitude to $\frac{dx_r}{dM}$. The vector ZW bears a similar relation to the X_s-axis. PW is therefore the sum of PZ and ZW.

The vector PV is obtained by multiplying the terms PW by the scalar quantity $-x_r$; that is,

$$PV = -x_r \sum_{i=1}^{n} \frac{\partial a_i}{\partial M} . \qquad (2:2:31)$$

The negative sign reverses the sense, and the size of x_r alters the magnitude. This vector is introduced here for convenience although it will not be used until later in the problem.

Variation (ii). The second of the simple variations is applied to the second basic situation. Here we allow C to vary in the equation of (2:2:27). In this case the locus of P can easily be shown to be the same expansion path as before.

One way of showing this is to note that, by differential geometry and under the conditions holding here for

restricted maximum and minimum values, the locus of the maximum for one function with respect to the value of a second is the same as the locus of the minimum of the second with respect to the first. Hence the locus of the maximum of u in the first situation corresponds, point by point, to the locus of the minimum of M in the second.

Another way of showing that the locus of P is the same expansion path as before is to find $\frac{dx_s}{dC}$ from (2:2:26) by the methods previously used. A comparison of the result with (2:2:29) yields

(2:2:32) $$\frac{dx_s}{dM} = \lambda \frac{dx_s}{dC}, \quad \text{or} \quad \frac{\partial \alpha_s}{\partial M} = \lambda \frac{\partial \beta_s}{\partial C}, \qquad s = 1, \ldots, n.$$

If the vector PF (not shown in Figure 2:2:3) is the vector sum of the derivatives with respect to C, then

(2:2:33) $$PW = \lambda \cdot PF .$$

Since the directions of the tangents coincide everywhere, the curves also coincide and the two loci are the same.

We note that, as a consequence of (2:2:32), the value of λ is $\frac{dC}{dM}$.

This variation seems to have mathematical significance only. The second relation in (2:2:32) will be used at a later point. On account of the non-measurability of utility, the equations of (2:2:32) and (2:2:33) appear to have no useful economic interpretation.

Variation (iii). The third of the simple variations applies also to the second basic situation in which utility is constant. In this variation we permit one of the prices, say p_r, to change. The locus of the equilibrium point P will be a one-dimensional curve on the indifference surface $u = C$. In the two-dimensional diagram, this curve is of necessity the indifference curve itself; it is the curve PE in Figure 2:2:3.

The values of the $\frac{dx_s}{dp_r}$ can be found from (2:2:26) as before. These values are

$$\frac{dx_s}{dp_r} = \frac{\partial \beta_s}{\partial p_r} = \frac{\lambda U_{rs}}{U} = x_{rs} \cdot \qquad s = 1,\ldots,n, \tag{2:2:34}$$

where U_{rs} is the cofactor of u_{rs} in U as defined above. The usual designation of the value of each derivative is with the symbol x_{rs}.

The vector PS of Figure 2:2:3 is defined as the sum of the components in (2:2:34), or

$$PS = \sum_{i=1}^{n} \frac{\partial \beta_i}{\partial p_r} = \sum_{i=1}^{n} x_{ri} \cdot \tag{2:2:35}$$

This vector lies along the tangent to the locus of P. The tangent line must lie in the plane tangent to the surface $u = C$ at P. Because M is a minimum at P, the tangent plane is also the budget plane. In Figure 2:2:3 the point S lies on the budget line through P determined by the prices p_r and p_s and the income M. It does

not lie on the curve PE as it is often pictured. The projection of PS upon the X_s-axis is represented by x_{rs}, and its projection upon the X_r-axis is represented by x_{rr}.

Regarded as vectors, x_{rs} and x_{sr} are not equal. While they have the same magnitudes, the first is parallel to the X_s-axis and the second is parallel to the X_r-axis.

The quantity M does not appear in the expressions for the x_{rs} since M is not a parameter in this problem.

The results of the three previous variations may be regarded as basic since the results of other types of variations are expressible in terms of them or by means of them.

Variation (iv). For the fourth type of variation we return to the first basic situation and allow one price, say p_r, to change. Under these conditions, the locus of the equilibrium point P will again be a one-dimensional curve (not shown) whose tangent in Figure 2:2:3 is the line PQ.

The values of the $\frac{dx_s}{dp_r}$ can be found from equations (2:2:24) as before. They are

$$\frac{dx_s}{dp_r} = - x_r \frac{\lambda U_s}{U} + \frac{\lambda U_{rs}}{U}, \qquad s = 1,\ldots,n. \tag{2:2:36}$$

With the use of previous relations, this reduces to

$$(2:2:37)\qquad \frac{dx_s}{dp_r} = \frac{\partial \alpha_s}{\partial p_r} = - x_r \frac{\partial \alpha_s}{\partial M} + \frac{\partial \beta_s}{\partial p_r}$$

$$= - \lambda x_r \frac{\partial \beta_s}{\partial C} + \frac{\partial \beta_s}{\partial p_r} , \quad s = 1,\ldots,n.$$

The above equation, together with (2:2:32), shows the relation between the derivatives of the α and β functions with respect to corresponding parameters. The familiar form of (2:2:37) is given by equation (2:2:23). However, the form of (2:2:23) fails to make clear that two separate functions of two different sets of parameters are involved, and that the "fundamental equation" is a combination of the more basic variations (i) and (iii).

If the vector PQ is defined as the sum of the simple vectors of (2:2:37), we have

$$(2:2:38)\qquad PQ = \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial p_r} = - x_r \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial M} + \sum_{i=1}^{n} \frac{\partial \beta_i}{\partial p_r} .$$

We have, therefore, as shown in Figure 2:2:3, that

$$(2:2:39)\qquad PQ = PV + PS .$$

Variation (v). In this variation we will permit several parameters to vary. If two or more parameters vary independently, the combined effect is found by the vector sum of their separate effects. If two or more parameters vary in such a way that all of the variation is dependent upon only one, then the derivatives of x_s with respect to the independent parameter may be found by the methods used above. In certain cases, the derivatives may be

found directly from the α and β functions themselves.

We now let a group of m prices change in such a way that they retain the same ratio with each other. For convenience in notation, we may number the goods whose prices vary proportionately so that they are the first m goods. Any one of these m prices, say p_r, may be chosen as the independent one.

If the income is fixed, as in the first basic situation, we have

(2:2:40)
$$\frac{dx_s}{dp_r} = \frac{\partial \alpha_s}{\partial p_1}\frac{dp_1}{dp_r} + \cdots + \frac{\partial \alpha_s}{\partial p_m}\frac{dp_m}{dp_r}$$
$$= \frac{\partial \alpha_s}{\partial p_1}\frac{p_1}{p_r} + \cdots + \frac{\partial \alpha_s}{\partial p_m}\frac{p_m}{p_r}$$
$$= \frac{1}{p_r}\sum_{i=1}^{m} p_i \frac{\partial \alpha_s}{\partial p_i}, \qquad s = 1,\ldots,n.$$

We may now substitute for the partial derivatives appearing in the last sum from the equations derived in the previous work. This is the key equation for the treatment of changes in value when the prices of a group of goods change proportionately.

With a fixed level of utility, as in the second basic situation, β replaces α in the above equations.

Variation (vi). In this type of variation we permit the price of one good, say p_r, to change and at the same time we adjust (or compensate) the income of the individual

so that he is always able to purchase the original amounts at the new prices. We propose to show that this "compensated variation in price" is just sufficient to allow the individual a constant level of utility.

Since the income is involved as a parameter, the α-form for x_s must be used. We have that $M = \sum_{i=1}^{n} p_i x_i$, from which we obtain that $\frac{\partial M}{\partial p_r} = x_r$. Using this relation and (2:2:25), we have

$$(2:2:41) \qquad \frac{dx_s}{dp_r} = \frac{\partial \alpha_s}{\partial p_r} + \frac{\partial \alpha_s}{\partial M}\frac{\partial M}{\partial p_r} = \frac{\partial \alpha_s}{\partial p_r} + x_r \frac{\partial \alpha_s}{\partial M}, \qquad s = 1,\ldots,n.$$

If PT (not shown) is the vector sum of the simple vectors of (2:2:41), we have

$$(2:2:42) \qquad PT = \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial p_r} + x_r \sum_{i=1}^{n} \frac{\partial \alpha_i}{\partial M}.$$

By (2:2:31), (2:2:38) and (2:2:39), this reduces to

$$(2:2:43) \qquad PT = PQ - PV = PS.$$

Therefore, the point T coincides exactly with the point S, not differing from it by infinitesimals of any order, as most of the literature states.

As a consequence of (2:2:43), the conditions of Variation (vi) are shown to produce a rate of change in the equilibrium point equivalent to that produced by the conditions of Variation (iii). This is the significance of the "compensation" of income to allow for variations in the price of a commodity.

PART III

PRODUCTION AND CONSUMPTION IN A STATIC ECONOMY

CHAPTER 1

A MATHEMATICAL MODEL

The primary purpose of this chapter is to combine production and consumption into a mathematically sound model of a closed static economy; and the secondary purpose is to make the model as realistic as possible as far as the economic aspects are concerned. It has very often been the custom in texts of econometrics to advance the arguments by economic reasoning; here the arguments will be advanced by mathematical reasoning. The economics will appear only as a basis for making the essential assumptions and as a field for interpreting the results.

A. Quantities Involved and Notation.

There are m individuals in the economy. The individuals will be identified by a left superscript.

There are n commodities in the economy, identified by X's with a right subscript.

There are ν variable factors of production in the economy, identified by Y's with a left subscript.

There are s fixed factors of production in the economy, identified by Z's with a left subscript.

For this static model we will assume that the number of persons in the economy does not change. We will also assume that the fixed factors of production remain constant in amounts and distribution. These fixed factors are constant in the sense that the amounts used in each production function are known.

The economy is to be a closed economy; i.e., there will be no transactions other than within the economy, and the consumption during a given period is limited to the production in that period plus any stocks of commodities which might have been on hand at the beginning of the period.*

This static closed model will be one in which production is equal to consumption during a given period. The analysis will take place after the economy has reached this point; i.e., the stocks and inventories at the end of a period will exactly equal those at the beginning of the period. Thus, stocks of commodities are classed among the Z's, or fixed capital assets, of the system.

Therefore, we are concerned with an economy which has reached a stationary population point and which has accumulated a fixed quantity and distribution of capital assets. We will consider such things as machinery, factories, etc.,

*If a particular commodity is perishable, then that portion of it which perishes will be considered as consumed by the individual having it in his possession at the time.

as being fixed factors. This, in effect, is an assumption that such capital assets have been expanded to the optimum point in the static economy, and having reached this point, that they are kept in a constant state of repair. In that sense they become constant known quantities.

The problem is one of determining the equilibrium point at which the economy has "settled down;" i.e., of determining the amounts of the n X's and the $n - 1$ price ratios of these commodities in terms of the fixed capital assets and their distribution. The solution will also embody certain known production and utility functions.

Certain other assumptions will become necessary in order to establish mathematical existence, continuity, uniqueness, etc. These assumptions will be introduced where they are required.

The following notation will be employed:

$X_1,\ldots,X_j,\ldots,X_n$ will denote the total amounts of of the n commodities consumed.

$\overline{X}_1,\ldots,\overline{X}_j,\ldots,\overline{X}_n$ will denote the total amounts of the n commodities produced, and will equal the amounts consumed. *

${}^iX_j$ will denote the amount of the j th commodity consumed by the i th individual.

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* The bars will be associated with the supply side of the problem.

${}_1Y, \ldots, {}_kY, \ldots, {}_\nu Y$ will denote the total amounts of the variable factors of production, and will be equal to the total amounts supplied.

${}_kY_j$ will denote the total amount of the k th factor used in the production of the j th commodity.

${}^i_kY_j$ will denote the amount of the k th factor used by the i th individual in the production of the j th commodity.

${}^i_k\bar{Y}$ will denote the amount of the k th factor supplied by the i th individual.

${}^i_hZ_j$ will denote the amount of the h th fixed factor possessed by the i th individual and used in the production of the j th commodity.

${}^if_j({}^i_1Y_j, \ldots, {}^i_kY_j, \ldots, {}^i_\nu Y_j, {}^i_1Z_j, \ldots, {}^i_hZ_j, \ldots, {}^i_\varepsilon Z_j)$ will denote the production function of the i th individual for the j th commodity.

${}^iU(X_1, \ldots, X_n, {}_1\bar{Y}, \ldots, {}_\nu\bar{Y})$ will denote the utility function for the i th individual.

${}^iU_j$ will denote the marginal utility of the j th commodity and ${}^i_kU$ the marginal utility of the k th factor to the i th individual. These are partial derivatives, and those with a left subscript will be opposite in sign from those with a right subscript.

P_j will denote the demand price of the j th commodity.

$\bar{P}_j$ will denote the supply price of the j th commodity.

${}^i\bar{P}_j$ will denote the average cost of production of the j th commodity for the i th individual.

${}_k\pi$ will denote the demand price of the k th variable factor of production.

${}^i\mu$ will denote the marginal utility of the numeraire to the i th individual.

B. Production Equations.

(a) Amounts Produced:

The total amounts of the commodities produced equal the sum of the amounts produced by the individuals, or

(3:1:1) $$\bar{X}_j = \sum_{i=1}^{m} {}^i\bar{X}_j \; ; \qquad j = 1,\ldots,n. \quad \text{(n equations)}$$

(b) Production Functions:

Each individual has a unique production function, involving the fixed and variable factors of production, for each commodity, or

(3:1:2) $$ {}^i\bar{X}_j = {}^if_j({}^i_1Y_j,\ldots,{}^i_\nu Y_j,\ {}^i_1Z_j,\ldots,{}^i_\varepsilon Z_j); \quad i = 1,\ldots,m. \quad j = 1,\ldots,n. \qquad \text{(mn equations)}$$

We assume here that the factors of production are

infinitely divisible, that all combinations of the factors are usable, and that the producer always combines the factors so as to obtain the maximum number of units of output which his technical knowledge permits. This assumption provides the existence, continuity and uniqueness necessary for mathematical manipulation.*

(c) Use of Factors for Different Commodities:

Each of the m individuals may use each of the ν factors in the production of each of the n commodities, or

(3:1:3) $$ {}_kY_j = \sum_{i=1}^{m} {}_iY_j; \qquad \begin{matrix} j = 1,\ldots,n. \\ k = 1,\ldots,\nu. \end{matrix} \quad (n\nu \text{ equations}) $$

(d) Total Use of Factors:

The total amount of the k th factor used equals the sum of the amounts used in producing each of the n commodities, or

(3:1:4) $$ {}_kY = \sum_{j=1}^{n} {}_kY_j; \qquad k = 1,\ldots,\nu. \quad (\nu \text{ equations}) $$

(e) Supply of Factors:

The total amounts of the factors supplied equals the sum of the amounts supplied by the individuals, or

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*The major portion of the analysis would hold with close approximation if our quantities changed by finite steps. The difficulties could be overcome by the use of the Calculus of Finite Differences. However, the results would be similar and a relaxation of the assumptions would hardly compensate for the increased complexity which would result.

(3:1:5) $$ {}_k\overline{Y} = \sum_{i=1}^{m} {}_k^i\overline{Y}; \qquad k = 1,\ldots\nu. \quad (\nu \text{ equations}) $$

(f) Cost of Production:

Each individual has an average cost of production for each commodity which he produces and this cost depends upon the prices of the factors used in the production, or

(3:1:6) $$ {}^i\overline{P}_j\,{}^i\overline{X}_j = \sum_{k=1}^{\nu} {}_k\pi\,{}_k^iY_j; \qquad \begin{matrix} i = 1,\ldots,m. \\ j = 1,\ldots,n. \end{matrix} \quad (mn \text{ equations}) $$

(g) Marginal Productivity:

We first assume that there is only one producer and that he produces a single commodity. For convenience we choose the first producer and the first commodity. The problem which we wish to investigate is one of minimizing the cost of production for a given quantity of production.

We have given the production function

(i) $$ {}^1\overline{X}_1 = {}^1f_1({}_1^1Y_1,\ldots,{}_k^1Y_1,\ldots,{}_\nu^1Y_1,{}_1^1Z_1,\ldots,{}_h^1Z_1,\ldots,{}_s^1Z_1). $$

The per unit prices of the Y's used in the production of ${}^1\overline{X}_1$ are ${}_1\pi,\ldots,{}_k\pi,\ldots,{}_\nu\pi$, and the cost of production is ${}^1\overline{P}_1\,{}^1\overline{X}_1$; hence,

(ii) $$ {}^1\overline{P}_1\,{}^1\overline{X}_1 = {}_1\pi\,{}_1^1Y_1 + \ldots + {}_k\pi\,{}_k^1Y_1 + \ldots + {}_\nu\pi\,{}_\nu^1Y_1. $$

We assume, in this case, that all amounts of the factors are used. This is necessary in order for the

equation (ii) to have uniqueness.

We now wish to minimize ${}^1F_1$ subject to the restraining condition of (i). Using the previously developed methods, we have

$$\text{(iii)} \qquad \frac{1}{1^\pi}\frac{\partial({}^1f_1)}{\partial({}^1_1Y_1)} = \ldots = \frac{1}{k^\pi}\frac{\partial({}^1f_1)}{\partial({}^1_kY_1)} = \ldots = \frac{1}{\nu^\pi}\frac{\partial({}^1f_1)}{\partial({}^1_\nu Y_1)} \,.^*$$

Extending equation (iii) to the case of m individuals and n commodities, we have

$$\text{(3:1:7)} \qquad \frac{1}{1^\pi}\frac{\partial}{\partial({}^1_1Y)}\,{}^1f_j = \ldots = \frac{1}{k^\pi}\frac{\partial}{\partial({}^1_kY)}\,{}^1f_j = \ldots = \frac{1}{\nu^\pi}\frac{\partial}{\partial({}^1_\nu Y)}\,{}^1f_j \;;$$

$$i = 1,\ldots,m. \qquad j = 1,\ldots,n. \qquad (mn\nu - mn \text{ equations})$$

We note that identical results would have been attained had we approached the problem by regarding the cost as fixed and maximizing output subject to the cost restriction.

C. Consumption Equations.

(a) Amounts Consumed:

The total amounts of the commodities consumed equal the sum of the amounts consumed by the individuals, or

$$\text{(3:1:8)} \qquad X_j = \sum_{i=1}^{m} {}^1X_j; \qquad j = 1,\ldots,n. \quad (n \text{ equations})$$

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*For the mathematics see p. 18 ff., and for the economics see p. 57 ff.

(b) Marginal Utilities:

We assume that there is only one individual, say the first, who at a given time has an income

$$(i) \qquad {}^1M = \sum_{j=1}^{n} P_j {}^1X_j - \sum_{k=1}^{\nu} k^{\pi} {}^1\overline{Y}_k ,$$

and who spends his entire income for the n available commodities. Assume also that the individual has a known utility function

$$(ii) \qquad {}^1U({}^1X_1, \ldots, {}^1X_n, {}^1\overline{Y}_1, \ldots, {}^1\overline{Y}_{\nu}),$$

and that the individual maximizes his utility subject to his budget restriction.

We form the expression

$$(iii) \qquad {}^1U - {}^1\mu\left[\sum_{j=1}^{n} P_j {}^1X_j - \sum_{k=1}^{\nu} k^{\pi} {}^1\overline{Y}_k\right] .$$

From the necessary conditions for a maximum, we have that

$$(iv) \qquad \frac{{}^1U_1}{P_1} = \ldots = \frac{{}^1U_j}{P_j} = \ldots = \frac{{}^1U_n}{P_n} = -\frac{{}^1U}{1^{\pi}} = \ldots = -\frac{{}^1U}{k^{\pi}} = \ldots$$
$$-\frac{{}^1U}{\nu^{\pi}} = {}^1\mu .$$

Extending equation (iv) to the case of m individuals, we have

$$(3:1:9) \qquad \frac{{}^iU_j}{P_j} = -\frac{{}^iU}{k^{\pi}} = {}^i\mu ; \qquad \begin{matrix} i = 1, \ldots, m. \\ j = 1, \ldots, n. \\ k = 1, \ldots, \nu. \end{matrix}$$

(mn + mν equations)

D. Perfect Competition Equations.

We assume that the economy operates under conditions of perfect competition. This means that

(3:1:10) $P_j = \overline{P}_j; \qquad j = 1,\ldots,n.$ (n equations)

Also under this assumption, no producer operates on a scale to affect $\overline{P}_j$.

E. Total Income Equations.

Each individual's total income equals his total expenditure. For the i th individual's total income we have

(i) Income from supply of factors equals $\sum_{k=1}^{\nu} k^{\pi}\, {}^{i}\overline{Y}$, and

(ii) Income from production equals $\sum_{j=1}^{n} (P_j - {}^{i}\overline{P}_j)\, {}^{i}\overline{X}_j$.

For the i th individual's total expenditure we have

(iii) $\sum_{j=1}^{n} P_j\, {}^{i}X_j$.

Thus, for the m individuals we have

(3:1:11) $$\sum_{k=1}^{\nu} k^{\pi}\, {}^{i}\overline{Y} + \sum_{j=1}^{n} (P_j - {}^{i}\overline{P}_j)\, {}^{i}\overline{X} = \sum_{j=1}^{n} P_j\, {}^{i}X_j;$$

$$i = 1,\ldots,m. \quad (m - 1 \text{ equations})$$

Since the total of the left expressions of (3:1:11) equals, from the cost of production equations, the total of the right expressions when all incomes are added

together, (3:1:11) actually yields only m - 1 independent equations.

In disposing of his product, each producer can pay his suppliers only with his product, and that portion of his product which is not needed to pay his suppliers is his profit (or entrepreneural wage). These profits are the sources of the M's, or incomes, p. 148, with which the producers maximize their utilities in the exchange arena.

The value of the product paid to the suppliers is the source of the M's with which they enter the exchange arena. Therefore, the entire production enters the exchange and may be regarded as consumed.

F. Maximization of Profit Equations.

We are able to express the $^{i}\overline{P}_{j}$ as functions of the $^{i}\overline{X}_{j}$ and a given set of parameters, or

$$(i) \qquad {}^{i}\overline{P}_{j} = {}^{i}{}_{j}(\,{}^{i}\overline{X}_{j}, {}_{1}{}^{i}\pi, \ldots, {}_{\nu}{}^{i}\pi, {}_{1}{}^{i}Z_{j}, \ldots, {}_{s}{}^{i}Z_{j}); \qquad \begin{matrix} i = 1,\ldots,m. \\ j = 1,\ldots,n. \end{matrix}$$

In order to do this we use (3:1:5), (3:1:6) and (3:1:7). From this system of $mn\nu + mn$ equations we eliminate the $mn\nu$ quantities $^{i}_{k}Y_{j}$ and obtain the relation (i).

We define the cost of production in the following manner:

$$(ii) \qquad {}^{i}C_{j} = {}^{i}\overline{P}_{j}\,{}^{i}\overline{X}_{j}; \qquad \begin{matrix} i = 1,\ldots,m. \\ j = 1,\ldots,n. \end{matrix}$$

Here we introduce mn new variables and mn new equations. However, they will not be included in the general summary since they appear only in this particular section.

We now make the assumption that each producer maximizes his profits,

$$(\overline{P}_j - {}^{i}\overline{P}_j)\,{}^{i}\overline{X}_j = \overline{P}_j\,{}^{i}\overline{X}_j - {}^{i}C_j; \qquad \begin{matrix} i = 1,\ldots,m. \\ j = 1,\ldots,n. \end{matrix} \tag{iii}$$

Here the ${}^{i}C_j$ are functions of the ${}^{i}\overline{X}_j$, the π's and the Z's.

From the necessary conditions for a maximum, and noting that the derivatives with respect to the ${}^{i}\overline{X}_j$ of the π's are zero (since they are parameters) and of the $\overline{P}_j$ are zero (from the assumption of the preceding section), we have

$$\frac{d}{d({}^{i}\overline{X}_j)}(\overline{P}_j - {}^{i}\overline{P}_j)\,{}^{i}\overline{X}_j = \overline{P}_j - {}^{i}\overline{P}_j - {}^{i}\overline{X}_j \frac{d}{d({}^{i}\overline{X}_j)}{}^{i}\overline{P}_j = 0, \tag{iv}$$

or

$$\overline{P}_j = {}^{i}\overline{P}_j + {}^{i}\overline{X}_j \frac{d}{d({}^{i}\overline{X}_j)}{}^{i}\overline{P}_j; \qquad \begin{matrix} i = 1,\ldots,m. \\ j = 1,\ldots,n. \end{matrix} \tag{3:1:12}$$

(mn equations)

G. Solution.

(a) Number of Equations:

From the equations of (3:1:1) through (3:1:12) we have a total of $mn\nu + n\nu + m\nu + 3mn + 3n + 2\nu + m - 1$ equations.

(b) Number of Variables:

The system of equations of the preceding paragraph contains $mn\nu + 3mn + m\nu + n\nu + 3n + 2\nu + m$ variables of the following types:

$mn\nu$ of the type ${}^{i}_{k}Y_{j}$

mn of the type ${}^{i}\overline{X}_{j}$

mn of the type ${}^{i}X_{j}$

mn of the type ${}^{i}\overline{P}_{j}$

$m\nu$ of the type ${}^{i}_{k}\overline{Y}$

$n\nu$ of the type ${}_{k}Y_{j}$

n of the type X_{j}

n of the type $\overline{P}_{j}$

n of the type P_{j}

ν of the type ${}_{k}Y$

ν of the type ${}_{k}\pi$

m of the type ${}^{i}\mu$.

(c) Elimination:

In the summaries of the two preceding paragraphs the number of equations is found to be one less than the number of variables. However, each equation in the system is either homogeneous in the set of quantities ${}^{i}\overline{P}_{j}$, ${}_{k}\pi$,

P_j, $\overline{P}_j$ and $\frac{1}{1\mu}$, or does not contain them. The solution, therefore, will be homogeneous of degree zero in the above set. This statement cannot be made of any other set of variables, and in particular, the solution will not be homogeneous in the X_j.

We are able to eliminate all of the variables except the X_j and the P_j, thereby obtaining $2n - 1$ equations. From these latter equations it is possible for us to determine the n amounts consumed and the $n - 1$ price ratios associated with these amounts.

H. Comments.

In the preceding model an attempt has been made to be completely general. It is possible for each of the m individuals to supply and to consume each of the n commodities, to supply and to use each of the ν variable factors of production, and to possess a quantity of each of the s fixed factors of production. Obviously, this will be an extreme case. In all probability, one individual will produce several, but not all, of the commodities and there will be many commodities which are not consumed by some of the individuals, etc. Also, the commodities (X_j's) and the variable factors of production ($_kY$'s) have been kept artificially distinct. It is probable that the results of one productive process (a half-finished good) may enter as a variable factor of production in another

productive process.

None of the above considerations presents any serious analytical difficulties. When some of the quantities are not involved there will be a certain number of equations which will drop out of the system, but they will be replaced, exactly in number, by variables equated to zero. For example, if the first individual consumes none of the first commodity, then $^1U_1$ will be less than $P_1{}^1\mu$, and one equation will be deleted from (3:1:9). However, this lost equation will be replaced by the equation $^1x_1 = 0$.

The following chapter will illustrate other manipulations of this nature.

CHAPTER 2

SPECIAL CASES AND MODIFICATIONS

In this chapter we will investigate a few of the more important concepts related to a static economy operating under competition. In so doing, we will make use of the notation and equations of the model of the preceding chapter.

The primary purpose of this chapter is to demonstrate, by a few modifications of the model and by a few special cases, the value of mathematical analysis in a problem of this type. Frequently the analysis will bring to light conclusions and limitations that might otherwise remain obscure. It also will establish cause and effect relationships clearly and concisely.

A. The Effect of Joint Demand.*

Joint demand will occur when one commodity, X_j, is useful only with another commodity, X_{j+1}; e.g., belts and belt buckles.

We may, without loss of generality, designate the commodities as the j th and j+1 st, and select our units so that one unit of X_j is required with one unit of X_{j+1}.

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* See Alfred Marshall: "Principles of Economics," p. 381 and p. 385.

We now have m new equations

(3:2:1) $${}^{i}X_{j} = {}^{i}X_{j+1}; \qquad i = 1,\ldots,m.$$

In the marginal utility equations of (3:1:9) the pairs of terms

(3:2:2) $$\frac{{}^{i}U_{j}}{P_{j}} = \frac{{}^{i}U_{j+1}}{P_{j+1}}; \qquad i = 1,\ldots,m,$$

will be replaced by the single term

(3:2:3) $$\frac{{}^{i}U'_{j}}{P_{j} + P_{j+1}}; \qquad i = 1,\ldots,m.$$

In the above equation ${}^{i}U'_{j}$ represents the marginal utility of one unit of X_{j} and X_{j+1} together.

Therefore we have m equations deleted from the system. The remaining equations are unaffected and a solution may be attained.

The above modification of the model is the simple case of a linear relationship of joint demand. The problem of the more general functional relationship may be managed in a similar manner.

In the general case we do not have to select our units so that one unit of X_{j} is required with one unit X_{j+1}, but may relate them by

(3:2:4) $${}^{i}X_{j} = {}^{i}g({}^{i}X_{j+1}); \qquad i = 1,\ldots,m.$$

Now the pairs of terms of (3:2:2) will be replaced

by the single term

$$(3{:}2{:}5) \qquad \frac{{}^{i}U_{j} + {}^{i}U_{j+1}\ {}^{i}g'({}^{i}X_{j+1})}{P_{j} + P_{j+1}\ {}^{i}g'({}^{i}X_{j+1})}\ ; \qquad i = 1,\ldots,m.$$

B. The Effect of Joint Supply.*

Joint supply will occur when two commodities, $\overline{X}_{j}$ and $\overline{X}_{j+1}$, result from one productive process; e.g., beef and hides.

There are few, if any, cases in which the cost of production of two or more joint products is exactly the same as the cost of production for one of them alone. If the result of any productive process has a market value, it is almost certain to have some care and expense devoted to it, and this care and expense will, in general, vary directly with the demand for the product. Thus, even in the case of joint supply, we are justified in assigning separate production functions to the jointly supplied products.

Consider the case of beef and hides. Let $\overline{X}_{j-1}$ represent the total quantity of cattle produced in the economy. Part of $\overline{X}_{j-1}$ may go to the dairy industry, part for breeding purposes, and part to the meat packing industry. Let this latter quantity be designated by ${}_{k}Y$,

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*See Alfred Marshall: "Principles of Economics," p. 388.

which will enter as a variable factor of production for beef ($\overline{X}_j$) and for hides ($\overline{X}_{j+1}$). There will be a relationship between the quantities of beef and hides produced, but since each producer must, in his own accounting system, impute his individual production costs (which, in general, will vary with the individual), we treat beef and hides as separate commodities.

Should the case occur where one of the two jointly supplied products costs nothing to produce, yet has a market value, our system will not have a solution.*

Where it becomes possible and desirable to modify the proportions of the joint products, the producer must be able to ascertain what portion of the whole expense of production is assigned to each product. The cost of anything used for several purposes has to be defrayed by the fruits of all of them. If there is no rule by which the producer can determine the relative importance of the uses, or the proportions in which the total cost should be distributed among them, then our system of equations will have no solution.

In cases such as "beef and hides" where the assumption of infinite divisibility does not hold, we encounter

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*We may consider this condition as a threshold zone; e.g., sawdust may have a market value but one not sufficiently large to influence the technique or the cost of producing lumber (width of saw cut, protection of sawdust pile, etc.). In this case we have a discontinuity in our system of equations where right and left hand derivatives are not equal.

another difficulty. To produce one more hide without increasing the production of beef would involve the price of a whole steer, while to produce one less hide without decreasing the production of beef would involve only the price of a hide. Thus, the marginal productivity derivatives would have a point of discontinuity. The economic problem is not an impossible one but the mathematics of the preceding chapters is incapable of handling it.

C. The Effect of Composite Supply.*

Composite supply will occur when a want can be satisfied by either of two commodities, X_j or X_{j+1}; e.g., beef or pork. The units (weights of beef and pork) may be chosen so that X_j and X_{j+1} are perfect alternatives, unit for unit, for one or more individuals.

In this case the utility function of any individual, say the first, having this consumer preference will take the form

$$(3:2:6)\quad {}^1U\left[{}^1x_1,\ldots,({}^1x_j+{}^1x_{j+1}),\ldots,{}^1x_n,\ {}^1\bar{y},\ldots,{}^1\bar{y}\right].$$

There are two possibilities; i.e., either the per unit prices of the commodities are equal or they are not equal. If $P_j = P_{j+1}$, then the conditional equation of (3:1:9)

$$(3:2:7)\qquad \frac{{}^1U_j}{P_j} = \frac{{}^1U_{j+1}}{P_{j+1}}$$

* See Alfred Marshall: "Principles of Economics," p. 390.

becomes an identity and cannot be counted among the set of simultaneous equations to be solved. Therefore, we have one equation deleted from the system, but the expression ${}^1x_j + {}^1x_{j+1}$ may be replaced by the symbol ${}^1x'_j$, and we have one less variable. The system is still determinate as far as ${}^1x'_j$ is concerned, although it will be impossible to determine ${}^1x_j$ and ${}^1x_{j+1}$ from their sum alone.

Also, in the consumption equations of (3:1:8) the two equations

$$(3:2:9) \qquad x_j = \sum_{i=1}^{m} {}^ix_j \quad \text{and} \quad x_{j+1} = \sum_{i=1}^{m} {}^ix_{j+1}$$

may be replaced by the single equation

$$(3:2:10) \qquad x'_j = \sum_{i=1}^{m} {}^ix'_j ,$$

but the system is indeterminate for the economy as a whole.

In the second case, where P_j is not equal to P_{j+1}, say where $P_j < P_{j+1}$, for an individual, say the first, then that individual will consume only ${}^1x_j$, and ${}^1x_{j+1}$ will be set equal to zero. One equation will be lost from the marginal utility equations but will be replaced by ${}^1x_{j+1} = 0$.

D. Demand and Supply Functions.

In this section we will regard the economy as an open economy and examine the demand for and the supply of

a commodity in terms of aggregates.

(a) Aggregate Demand Functions:

Let ${}^{i}M$ denote the total expenditure of the i th consumer in a given time interval and assume the ${}^{i}M$ to be known.

We will make use of the following sets of equations:

$$(3:2:11) \qquad {}^{i}M = \sum_{j=1}^{n} P_j {}^{i}X_j; \qquad i = 1,\ldots,m. \quad (m \text{ equations})$$

$$(3:2:12) \qquad \frac{{}^{i}U_1}{P_1} = \ldots = \frac{{}^{i}U_j}{P_j} = \ldots = \frac{{}^{i}U_n}{P_n}; \qquad i = 1,\ldots,m. \quad (mn - m \text{ equations})$$

$$(3:2:13) \qquad X_j = \sum_{i=1}^{m} {}^{i}X_j, \qquad j = 1,\ldots,n. \quad (n \text{ equations})$$

From the above system of $mn + n$ equations we eliminate the mn quantities ${}^{i}X_j$. The result will be a set of n equations involving the quantities P_j, X_j and ${}^{i}M$. We solve these n equations separately for the quantities P_j, obtaining

$$(3:2:14) \qquad P_j = \phi_j(X_1,\ldots,X_j,\ldots,X_n,{}^{i}M); \quad j = 1,\ldots,n.$$

Thus we have the equations of the demand curves.

(b) Aggregate Supply Functions:

Consider the following sets of equations:

$$(3:2:15) \qquad {}^{i}\overline{X}_j = {}^{i}f_j({}^{i}_{k}Y_j, {}^{i}_{h}Z_j); \quad \begin{matrix} i = 1,\ldots,m. \\ j = 1,\ldots n. \\ k = 1,\ldots,y. \\ h = 1,\ldots,s. \end{matrix} \quad (mn \text{ equations})$$

(3:2:16) $\frac{1}{1^{\pi}} \frac{\partial}{\partial(\overset{1}{1}Y)} \overset{1}{1}f_j = \ldots = \frac{1}{k^{\pi}} \frac{\partial}{\partial(\overset{1}{k}Y)} \overset{1}{k}f_j = \ldots = \frac{1}{\nu^{\pi}} \frac{\partial}{\partial(\overset{1}{\nu}Y)} \overset{1}{\nu}f_j$;

$i = 1,\ldots,m.$
$j = 1,\ldots,n.$ ($mn\nu - mn$ equations)

(3:2:17) $\overset{1}{\overline{P}}_j = \overset{1}{\overline{P}}_j + \overset{1}{\overline{X}}_j \frac{\partial}{\partial(\overset{1}{\overline{X}}_j)} \overset{1}{\overline{P}}_j$; $i = 1,\ldots,m.$ $j = 1,\ldots,n.$ (mn equations)

(3:2:18) $\overset{1}{\overline{P}}_j \overset{1}{\overline{X}}_j = \sum_{k=1}^{\nu} {}_k{}^{\pi}\, \overset{1}{k}Y_j$; $i = 1,\ldots,m.$ $j = 1,\ldots,n.$ (mn equations)

(3:2:19) $\overline{X}_j = \sum_{i=1}^{m} \overset{1}{\overline{X}}_j$; $j = 1,\ldots,n.$ (n equations)

From the above system of $2mn + mn\nu + n$ equations we eliminate the $2mn + mn\nu$ quantities $\overset{1}{k}Y_j$, $\overset{1}{\overline{X}}_j$ and $\overset{1}{\overline{P}}_j$. The result is a set of n equations involving the quantities $\overline{P}_j$, $\overline{X}_j$, ${}_k{}^{\pi}$ and $\overset{1}{h}Z_j$. The quantities $\overset{1}{h}Z_j$ are known and we now make the assumption that the ${}_k{}^{\pi}$ are fixed quantities. We solve these n equations separately for the quantities $\overline{P}_j$, obtaining

(3:2:20) $\overline{P}_j = \psi_j(\overline{X}_1,\ldots,\overline{X}_j,\ldots,\overline{X}_n,\ {}_k{}^{\pi},\ \overset{1}{h}Z_j)$; $j = 1,\ldots,n.$

Thus we have the equations of the supply curves.

For a single commodity, say the first, we have for the demand equation

(3:2:21) $P_1 = \Phi_1(X_1)$,

and for the supply equation

(3:2:22) $$\bar{P}_1 = \Psi_1(\bar{X}_1).$$

We note that the derivation of these equations involves assumptions seldom explicitly stated. In the two equations above the quantities of the remaining $n - 1$ commodities consumed, $X_2,\ldots,X_n$, or produced, $\bar{X}_2,\ldots,\bar{X}_n$, are held constant only by adjusting their prices in such a manner as to hold them constant. Also, in the supply equations the ν variable factors of production must be distinct from the commodities produced.

Had we elected to solve for the X_j and the $\bar{X}_j$ instead of for the P_j and the $\bar{P}_j$, we would have obtained in place of the ϕ and Ψ functions of (3:2:14) and (3:2:20)

(3:2:23) $$X_j = \theta_j(P_1,\ldots,P_j,\ldots,P_n,\ {}^{i}M);\quad j = 1,\ldots,n,$$

and

(3:2:24) $$\bar{X}_j = \omega_j(\bar{P}_1,\ldots,\bar{P}_j,\ldots,\bar{P}_n,\ {}_{k}\pi,\ {}^{i}_{h}z_j);\quad j = 1,\ldots,n.$$

In place of (3:2:21) we could have constructed the demand equation as

(3:2:25) $$X_1 = \Theta(P_1)$$

by adjusting the quantities consumed, $X_2,\ldots,X_n$, so that the prices, $P_2,\ldots,P_n$, remained constant; and in place of

(3:2:22) we could have constructed the supply equation as

(3:2:26) $$\overline{X}_1 = \Omega_1(\overline{P}_1)$$

by adjusting the quantities $\overline{X}_2,\ldots,\overline{X}_n$ so that $\overline{P}_2,\ldots,\overline{P}_n$ remained constant.

E. Elasticities.

The concept of elasticity is one of importance in economics. Here we will consider certain elasticities in terms of the notation of the previous sections.

Elasticity is a ratio of the relative change in one quantity to a relative change in another, where it is understood that there is a functional relationship between the quantities. In economics it is a numerical expression usually calculated by the formula

(3:2:27) $$\text{Elasticity} = \frac{\dfrac{\text{change in quantity}}{\text{quantity}}}{\dfrac{\text{change in price}}{\text{price}}} = \frac{\dfrac{\Delta X}{X}}{\dfrac{\Delta P}{P}} = \frac{P}{X}\frac{\Delta X}{\Delta P}.$$

(a) Elasticity of Demand:

Here we are interested in the relationship between changes in the quantity of X_1 demanded and changes in the demand price of X_1.

We form the ratio

(1) $$\epsilon_1 = -\frac{\dfrac{P_1}{\Delta P_1}}{\dfrac{X_1}{\Delta X_1}} = -\frac{P_1}{X_1}\frac{1}{\dfrac{\Delta P_1}{\Delta X_1}}.$$

Taking the limit as $\Delta X_1 \to 0$, we have

$$\epsilon_1 = -\frac{P_1}{X_1}\frac{1}{\frac{\partial P_1}{\partial X_1}}, \tag{11}$$

or, from (3:2:21),

$$\epsilon_1 = -\frac{\Phi_1(X_1)}{X_1\Phi_1'(X_1)}, \tag{3:2:28}$$

where the prime denotes the derivative of $\Phi_1(X_1)$ with respect to X_1.

In general, $\Phi_1'(X_1)$ is negative, the quantity of X_1 demanded decreasing as the price increases, and therefore ϵ_1 will be positive.

Had we elected to use (3:2:25), where $X_1 = \Theta_1(P_1)$, we would have obtained

$$\epsilon_1' = \frac{P_1}{X_1}\frac{\partial X_1}{\partial P_1}. \tag{3:2:29}$$

We note that in obtaining ϵ_1 we assumed that the prices $P_2,\ldots,P_n$ were adjusted so as to hold $X_2,\ldots,X_n$ constant; while in obtaining ϵ_1' we assumed that the quantities $X_2,\ldots,X_n$ were adjusted so as to hold $P_2,\ldots,P_n$ constant. Hence ϵ_1 and ϵ_1' will not be equal except by rare coincidence. It is therefore important to know exactly what assumptions lie behind the evaluation of an elasticity.

(b) Elasticity of Supply:

Here we are interested in the relationship between changes in the quantity of $\overline{X}_1$ supplied and changes in the supply price of $\overline{X}_1$.

We form the ratio

$$\eta_1 = \frac{\dfrac{\overline{P}_1}{\Delta\overline{P}_1}}{\dfrac{\overline{X}_1}{\Delta\overline{X}_1}} = \frac{\overline{P}_1}{\overline{X}_1}\,\frac{1}{\dfrac{\Delta\overline{P}_1}{\Delta\overline{X}_1}} \,. \tag{i}$$

Taking the limit as $\Delta\overline{X}_1 \to 0$, we have

$$\eta_1 = \frac{\overline{P}_1}{\overline{X}_1}\,\frac{1}{\dfrac{\partial\overline{P}_1}{\partial\overline{X}_1}} \,, \tag{ii}$$

or, from (3:2:22),

$$\eta_1 = \frac{\Psi_1(\overline{X}_1)}{\overline{X}_1\Psi_1'(\overline{X}_1)} \,. \tag{3:2:30}$$

Had we elected to use (3:2:25), where $\overline{X}_1 = \Omega_1(\overline{P}_1)$, we would have obtained

$$\eta_1' = \frac{\overline{P}_1}{\overline{X}_1}\,\frac{\partial\overline{X}_1}{\partial\overline{P}_1} \,, \tag{3:2:31}$$

and as before, η_1' will not, in general, be equal to η_1.

(c) Elasticity of Cost:

We define the quantities $\overline{M}_1$ and ξ_1 as follows:

(i) $\bar{M}_1 = \bar{P}_1 \bar{X}_1 = \bar{X}_1 \bar{\Psi}_1(\bar{X}_1)$, and

(ii) $$\xi_1 = \frac{\bar{M}_1}{\bar{X}_1 \frac{\partial \bar{M}_1}{\partial \bar{X}_1}} .$$

We note that ξ_1 measures the ratio of the relative change in the cost of production to the relative change in output. From the preceding section we have that η_1 measures the relative change in price to the relative change in output. Since the two elasticities, ξ_1 and η_1, are computed under the same assumptions, we are able to compare their values.

We have

(iii) $$\frac{\partial \bar{M}_1}{\partial \bar{X}_1} = \bar{\Psi}_1(\bar{X}_1) + \bar{X}_1 \bar{\Psi}'_1(\bar{X}_1);$$

hence

(iv) $$\xi_1 = \frac{\bar{X}_1 \bar{\Psi}_1(\bar{X}_1)}{\bar{X}_1 \left[\bar{\Psi}_1(\bar{X}_1) + \bar{X}_1 \bar{\Psi}'_1(\bar{X}_1) \right]}, \quad \text{or}$$

(3:2:32) $$\xi_1 = \frac{\bar{\Psi}_1(\bar{X}_1)}{\bar{\Psi}_1(\bar{X}_1) + \bar{X}_1 \bar{\Psi}'_1(\bar{X}_1)} .$$

Using the above equation and (3:2:30), we obtain the relationship

(v) $$\frac{1}{\xi_1} - \frac{1}{\eta_1} = 1 + \frac{\bar{X}_1 \bar{\Psi}'_1(\bar{X}_1)}{\bar{\Psi}_1(\bar{X}_1)} - \frac{\bar{X}_1 \bar{\Psi}'_1(\bar{X}_1)}{\bar{\Psi}_1(\bar{X}_1)} = 1.$$

From the above equation we establish the following relationship between the elasticity of supply and the elasticity of cost:

(3:2:33) $$\eta_1 = \frac{\xi_1}{1 - \xi_1} \qquad \xi_1 = \frac{\eta_1}{1 + \eta_1} .$$

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VITAE

The author was born July 11, 1911, at Birmingham, Alabama. In 1927 he was graduated from Simpson High School of that city. He received the degree of Bachelor of Arts, with highest honors, from Birmingham-Southern College in 1931. He received the degree of Master of Arts from the University of Florida in 1933. In 1934 and 1935 he served as athletic coach and physics instructor at Butler County High School of Greenville, Alabama. From 1936 to 1938 he was employed as Physical Testing Engineer by the Pittsburgh Testing Laboratory. In 1938 he accepted a position as Instructor of Mathematics at The Clemson Agricultural College of Clemson, South Carolina. From 1943 to 1946 he served as an officer in the United States Naval Reserve, primarily in the Caribbean area. He was released to inactive duty with the rank of Senior Grade Lieutenant. After resuming his duties at Clemson College he was promoted to the rank of Associate Professor of Mathematics. In 1949 he was granted two years' leave of absence from Clemson College to pursue graduate studies at the University of Florida. He is a member of the Mathematical Association of America, the American Association of University Professors, Pi Kappa Alpha social fraternity, Omicron Delta Kappa and Kappa Phi Kappa honor societies.

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